MOTIVATION: Planar graphs

Thm: [de Fraysseix, Pach, Pollack '90, Schnyder '89]

PLANAR GRAPHS HAVE $\Theta(n) \times \Theta(n)$ 2D GRID DRAWINGS

$\Theta(n^2)$ volume

Q: [Felsner, Liotta, Wismath '01]

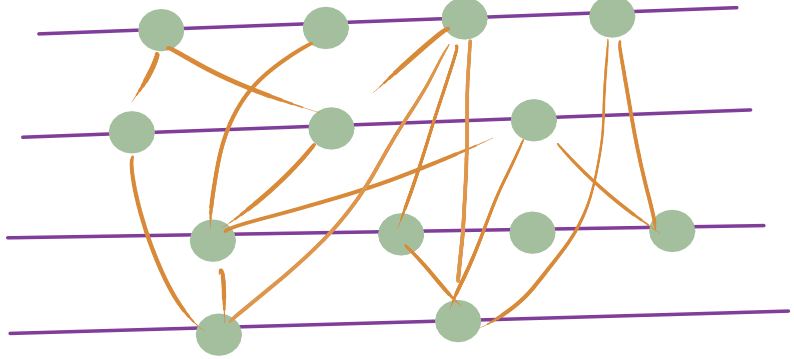
Can we do better in 3D?

→ $\Theta(n)$

→ $o(n^2)$

WHAT IS KNOWN ?

GRAPH FAMILY	VOLUME	REFERENCE
K_n, arbitrary	$\Theta(n^3)$	Eades, Cohen, Lin, Ruskey '96
$O(1)$ colourable	$\Theta(n^2)$	Pach, Thiele, Toth '97
$O(1)$ max degree	$\Theta(n^{3/2})$	D. & Wood '04
$O(1)$ outerplanar series-parallel	$\Theta(n)$	– Felsner, Liotta, Wismath '01 – Di Giacomo, Liotta, Wismath '02
$O(1)$ treewidth	$\Theta(n)$	– D., Morin, Wood '05 – Wiechert '18

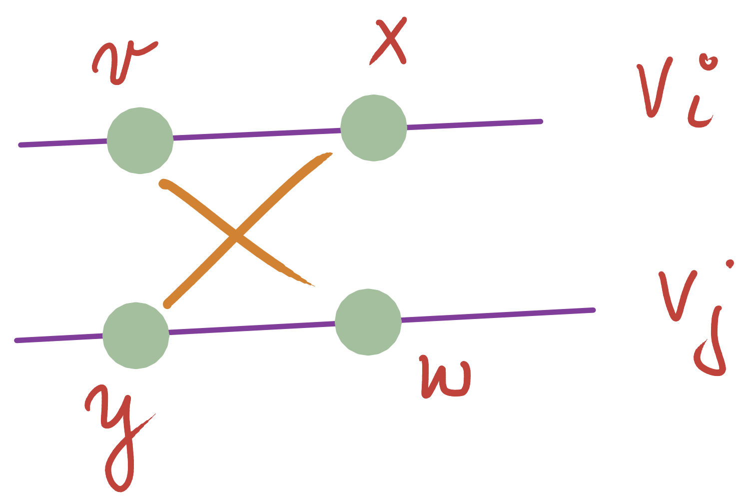


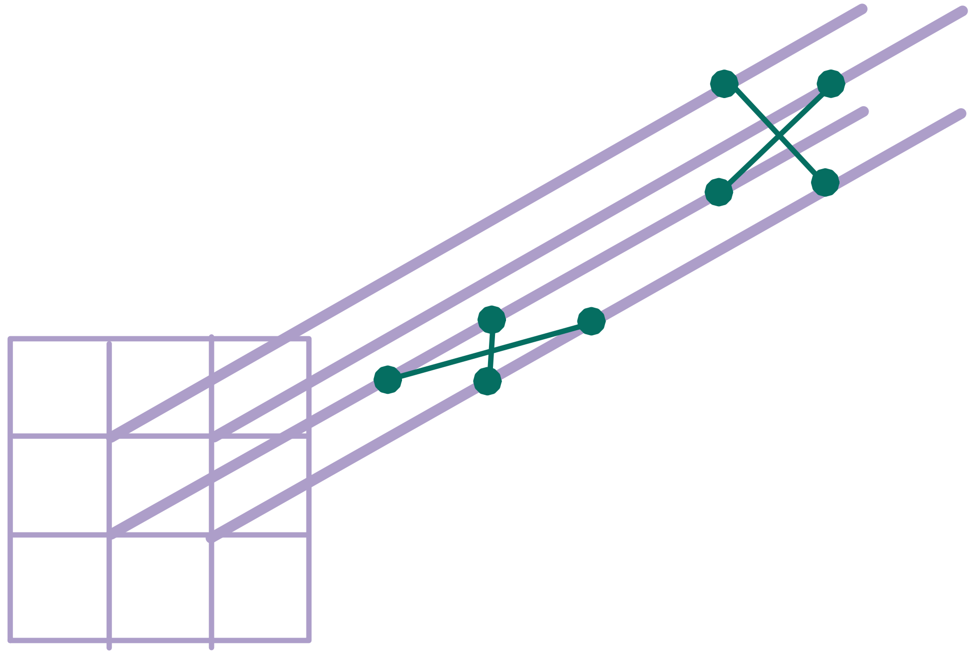
t-TRACK LAYOUT

- $\{V_1, V_2, \ldots, V_t\}$ – vertex colouring
- total order $<_i$ of each V_i (track)
- no X-crossing

Def: X-crossing

edges vw, xy
s.t $v <_i x$ and $y <_j w$





why track layouts ?

Thm: [D. & Wood '04 / D., Morin, Wood '05]
Every graph G with t-track layout has a 3D grid drawing in $O(t^2 \cdot n)$ volume.

$O(\chi^7 \cdot t \cdot n)$

Back to track number of planar graphs

Q1:

Do planar graphs have $\Theta(1)$ track num?

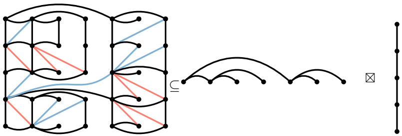
↳ $\Theta(n)$ 3D grid drawing

Why?

planar

$G \subseteq H \boxtimes P$

- H is a graph of treewidth at most 8
- Many problems are easy for H (or known)
- Extending a solution from H to $H \boxtimes P$ is sometimes easy



Q1:

Do planar graphs have $O(1)$ track num?

$\hookrightarrow$ $O(n)$ 3D grid drawing

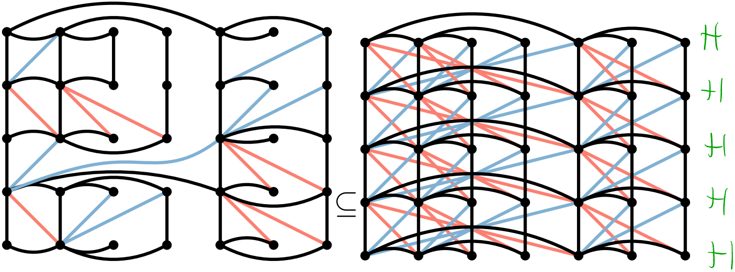
Th: [D., Morin, Wood 2005] [Wiechert 2016]

graphs of bounded treewidth have $O(1)$ track number

$tn(G) \leq (k+1)(2^{k+1}-2)^k$ where $k := tw(G)$

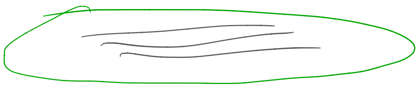
structure of planar graphs

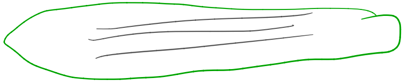
theorem [Dujmović, Joret, Micek, Morin, Ueckerdt, Wood '19]
every planar graph G is a subgraph of $H \boxtimes P$
for some graph H with treewidth $\leqslant 8$ and some path P

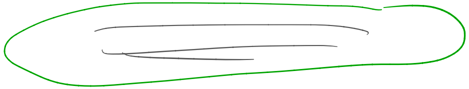


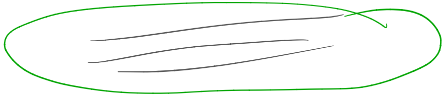
$t := tn(H)$

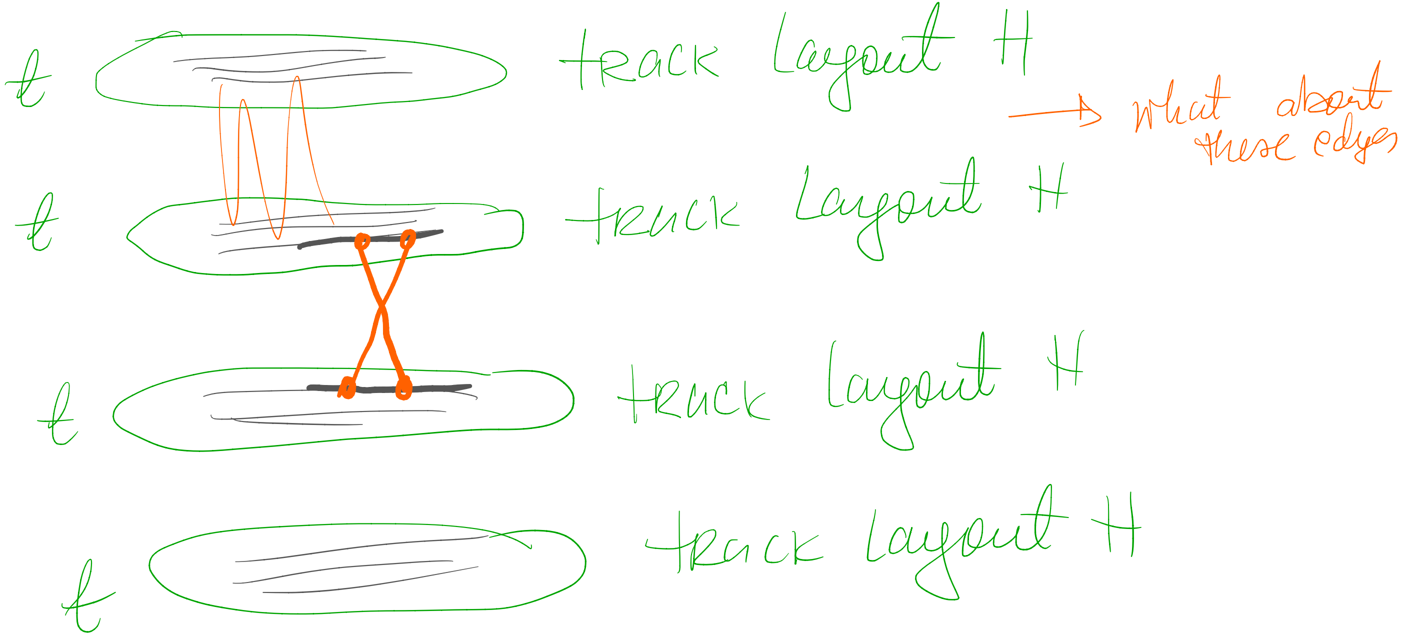
TRY obvious thing

t  track layout H

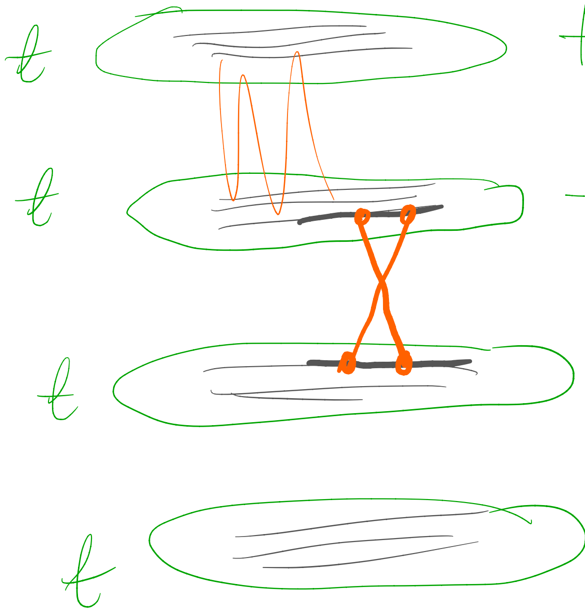
t  track layout H

t  track layout H

t  track layout H



t
track layout H
what about these edges
t
track layout H
t
track layout H
t
track layout H



track layout H

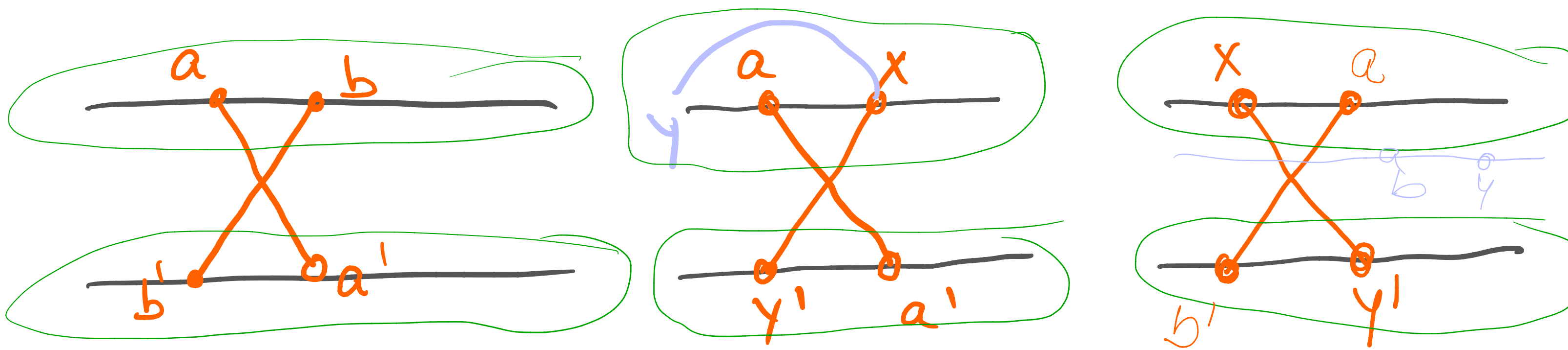
track layout H

track layout H

track layout H

→ what about these edges

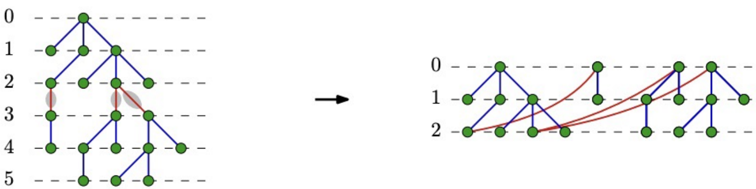
1. no X-crossings (need to show)
2. span $\leq 2t-1$



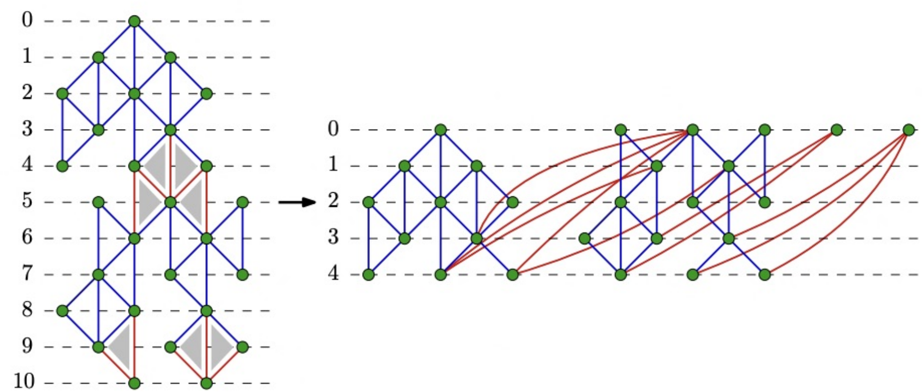
a
b
b'
a'
a
x
y
y'
a'
x
a
b
y
b'
y'

Span → tn

Thm/ track layout : $\{(V_i, <_i) : 1 \leq i \leq t\}$
with edge span s ⇒ $tn(G) \leq 2 \cdot s + 1$.



Images by Sergey Pupyrev



Track number product

$2\cdot 8+1$

(*) $tn(H \boxtimes P) \leq 2\cdot(2\cdot tn(H)-1)+1$
$\leq 4\,tn(H)-1$

Product structure $\Rightarrow$ $\forall G$ planar $G \subseteq H \boxtimes P$
$tw(H) \leq 8$

$\Rightarrow tn(H) \in \Theta(1)$

(*) $\Rightarrow$ planar graphs have $\Theta(1)$ track #

3D drawings of other graphs classes

(track number)

$\Theta(1)$ track # $\Rightarrow$ $O(n)$ 3D grid drawings

~~$\Leftarrow$~~

- What are good candidates?
- How many edges can $O(1)$-track graph have?
- How many edges can a graph in $O(n)$ volume have?

MIDPOINT

What are the candidates?

graphs with $\Theta(n)$ edges:

- bounded genus
- minor closed
- k-planar
- bounded degree

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Which of these families admit $P \boxtimes H$ product structure?

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Which of these families admit $P \boxtimes H$ product structure?

bounded degree

candidate for :

$O(1)$ track number ?

$O(n)$ 3D grid drawing ?

bounded degree

candidate for :

$\Theta(1)$ track number ? NO (counting)

$\Theta(n)$ 3D grid drawing ?

BOUNDED DEGREE

Q: [Pach, Thiele, Tóth, '99]

Does every bounded degree graph have $O(n)$ volume 3D grid drawing?

→ $tn \in \Omega(\sqrt{\Delta}\, n^{\frac{1}{2}-\frac{1}{\Delta}})$

→ $\exists$ bounded degree expanders with $\Theta(1)$ tn

COUNTING

Th [Bender & Canfield '78, Warmald '78, McKay '85]
Num. of labelled Δ-regular graphs is

$$\geq \left(\frac{n}{3\Delta}\right)^{\Delta n/2}$$

Q: Given a 3D grid of volume N, how many crossing-free graphs does it admit?

↳ $\Theta(c^N)$?

COUNTING via CROSSING LEMMA

[Ajtai, Chvátal, Newborn, Szemerédi '82]

	CROSSING LEMMA:	# GRAPHS:
in 2D:	$\Omega\left(\frac{m^3}{N^2}\right)$	$O(c^N)$

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	CROSSING LEMMA:	# GRAPHS:
in 2D:	$\Omega\left(\frac{m^3}{N^2}\right)$	$O(c^N)$

[D., Morin, Sheffer '13]

	CROSSING LEMMA:	# GRAPHS:
in 3D:	$\Omega\left(\frac{m^2}{N}\log\frac{m}{N}\right)$	?
in 4⁺D:	$\Omega\left(\frac{m^2}{N}\right)$	$\Omega(N^N)$

→ tight

CANDIDATES: families with P⊠H structure

- planar ✓

graphs with $\Theta(n)$ edges:

- bounded genus ✓
- minor closed
- k-planar ✓
- bounded degree

2 consecutive rows have bounded tw
thus $O(n_1+n_2)$ edges
$O(n_1+n_2+n_3 \ldots)$

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– minor closed ?
– k-planar ✓
– bounded degree NO

2 consecutive rows have bounded tw thus $\Theta(n_1+n_2)$ edges $\Theta(n_1+n_2+n_3 \dots)$

P⊠H graphs have $\Theta(\sqrt{n})$ separators

PRODUCT STRUCTURE THEORY

- variations
- generalizations (other classes)
- other applications

Variations

The Product Structure Theorem for Planar Graphs

Theorem (Dujmović-Joret-Micek-M-Ueckerdt-Wood 2019):
For every planar graph G, there exists a planar graph H of treewidth at most 8 and a path P such that G is a subgraph of $H \boxtimes P$.

orin

→ simple treewidth 6

[Ueckerdt, Wood, Yi, 2021]

Main theorem: Proof

Th: [D., Joret, Micek, Morin, Ueckerdt, Wood] 2019

For every BFS spanning tree T of a planar graph G, $\exists$ a partition $\mathcal{P}$ into vertical paths s.t. treewidth$(G/\mathcal{P}) \le 8$.

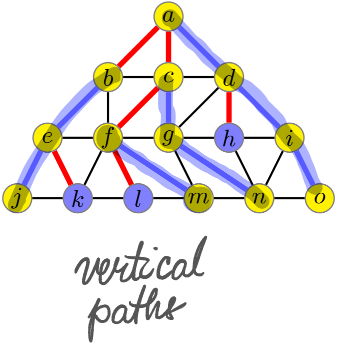
Proof: Partitioning Planar Graphs

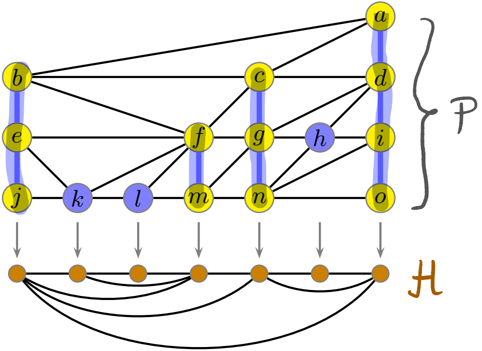
Key lemma. Suppose

- G^+ plane triangulation
- T rooted spanning tree of G^+ with root on outer-face
- cycle C partitioned into vertical paths $P_1, \ldots, P_k$, with $k \leq 6$
- G near-triangulation consisting of C and everything inside.

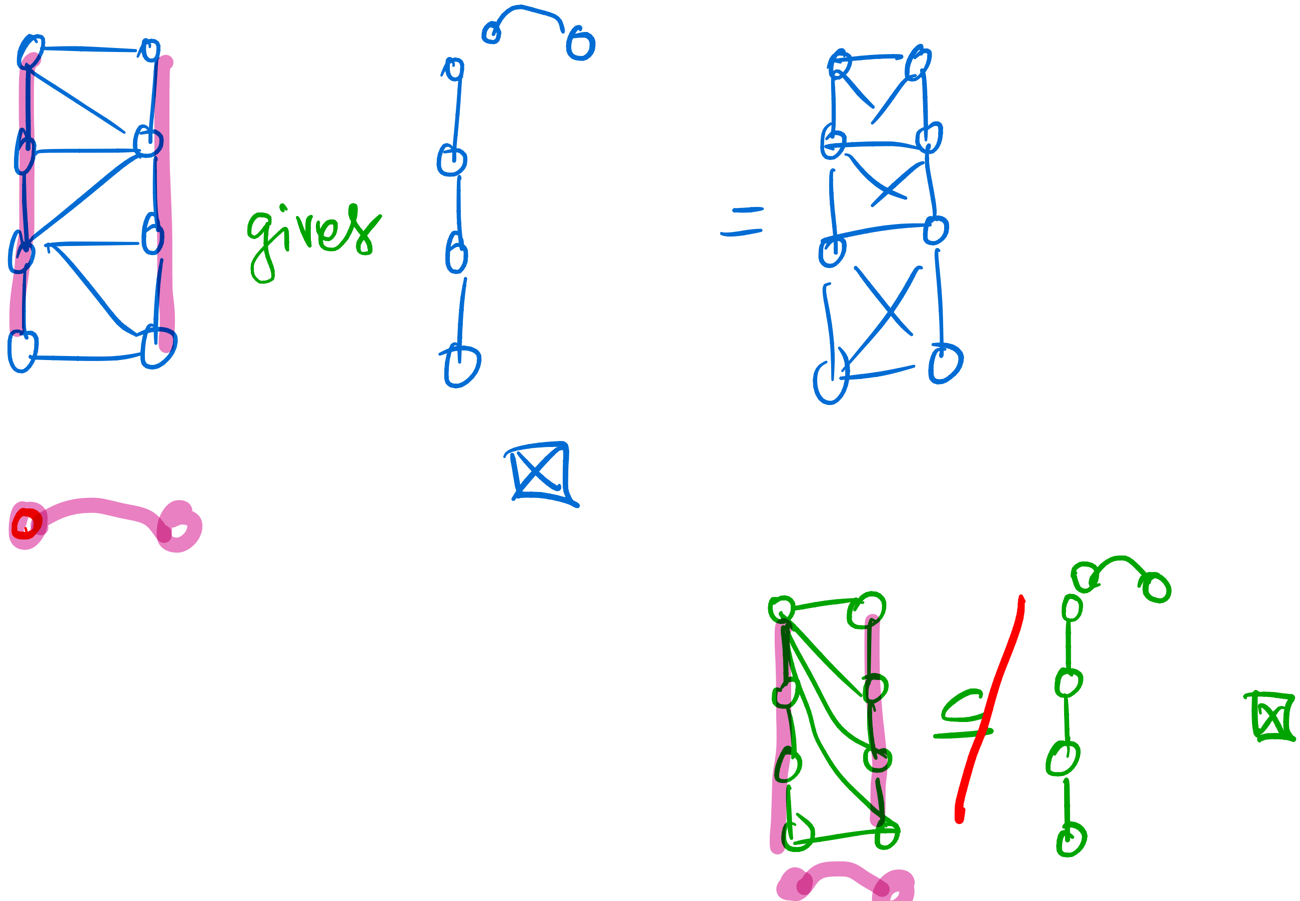
Then G has a partition $\mathcal{P}$ into vertical paths where $P_1, \ldots, P_k \in \mathcal{P}$ s.t. $= G/\mathcal{P}$ has a tree-decomposition in which every bag has size at most 9 and some bag contains all vertices corresponding to $P_1, \ldots, P_k$.

Equivalence :



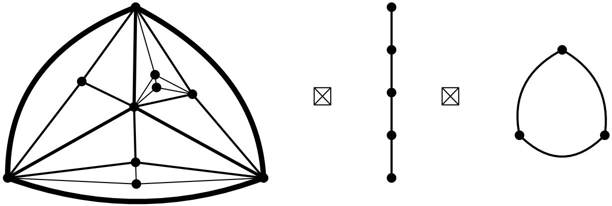


$G \subseteq H \boxtimes P$


gives

Second Version

Theorem (Dujmović-Joret-Micek-M-Ueckerdt-Wood 2019):
For every planar graph G there exists a planar graph H of treewidth at most 3 such that $G \subseteq H \boxtimes P \boxtimes K_3$.



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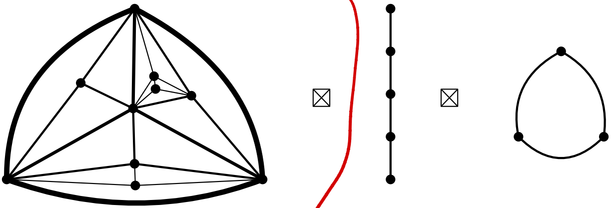

- planar
- tw 3
- MINOR of G

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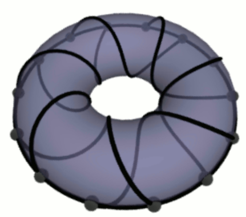
Generalizations

Generalizations

Similar* product structure theorems for

H minor

- graphs of bounded genus and apex-minor free graphs (Dujmović-Joret-Micek-M-Ueckerdt-Wood 2019);



*$G \subseteq H \boxtimes P$, only the treewidth of H changes

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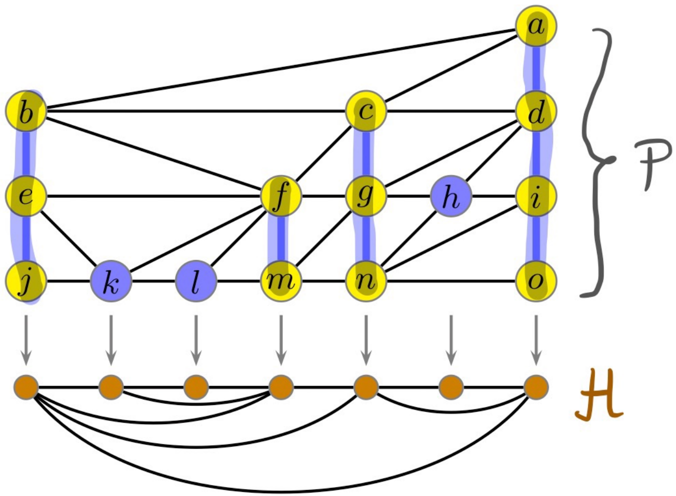


$G \subseteq H \boxtimes P$

$\downarrow$

if $|P| = const$

and $tw(H) = const$ $\Longrightarrow$ $tw(G) = const$

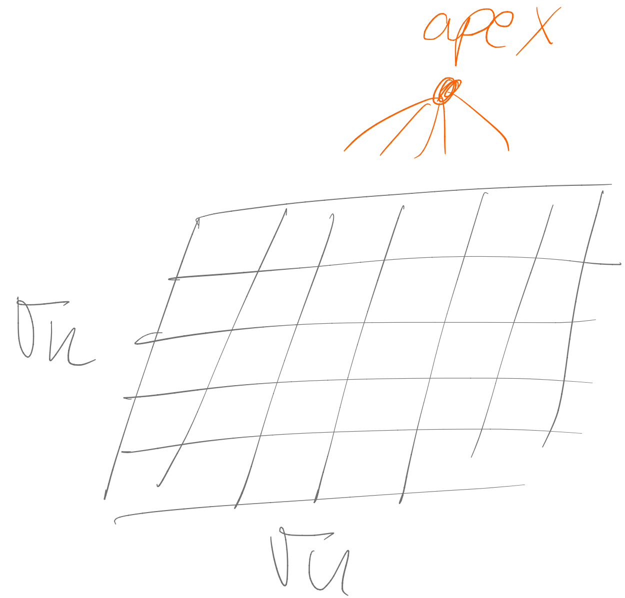
$G \subseteq H \boxtimes P$

if $|P| = const$

and $tw(H) = const$

$\Downarrow$

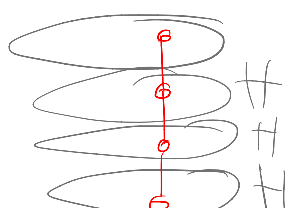
$tw(G) = const$



$\Downarrow$

$G \subseteq H \square P_3$

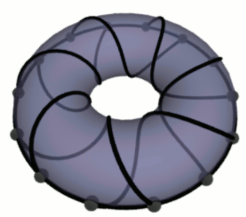
since $H \square P_{4+}$
has vert at
distance 3+



Generalizations

Similar* product structure theorems for

- graphs of bounded genus and apex-minor free graphs (Dujmović-Joret-Micek-M-Ueckerdt-Wood 2019);



*$G \subseteq H \boxtimes P$, only the treewidth of H changes

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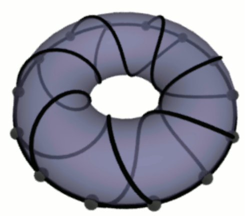

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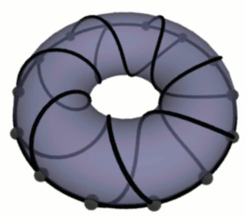
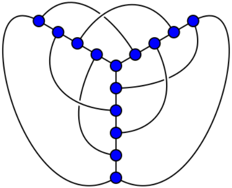
- bounded degree graphs that exclude a fixed graph as a minor (Dujmović-Esperet-M-Walczak-Wood 2020);
- k-planar graphs and (g, k)-planar graphs (Dujmović-M-Wood 2019).

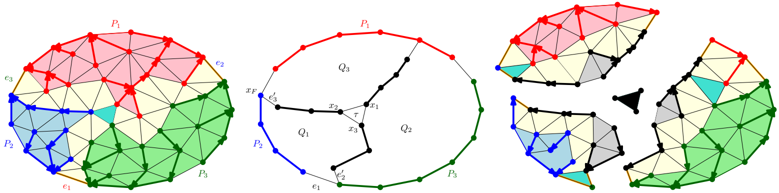




*$G \subseteq H \boxtimes P$, only the treewidth of H changes

Algorithmic Version

Theorem (Morin 2021): There exists an $O(n \log n)$ time algorithm that, given an n-vertex planar triangulation G finds H and P and the mapping $V(G) \rightarrow V(H \boxtimes P)$.



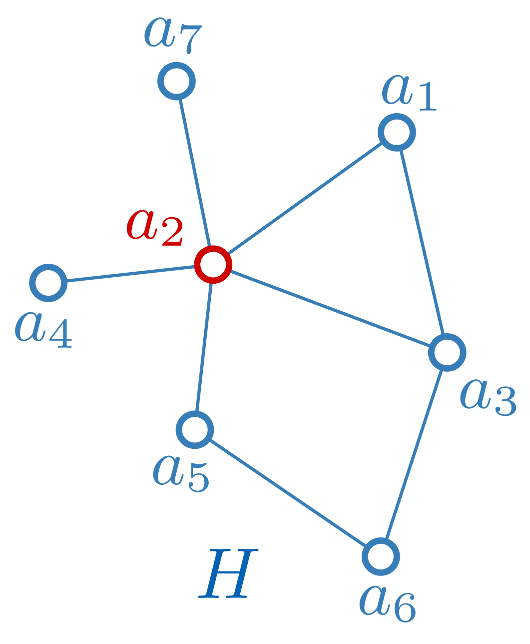
https://github.com/patmorin/lhp

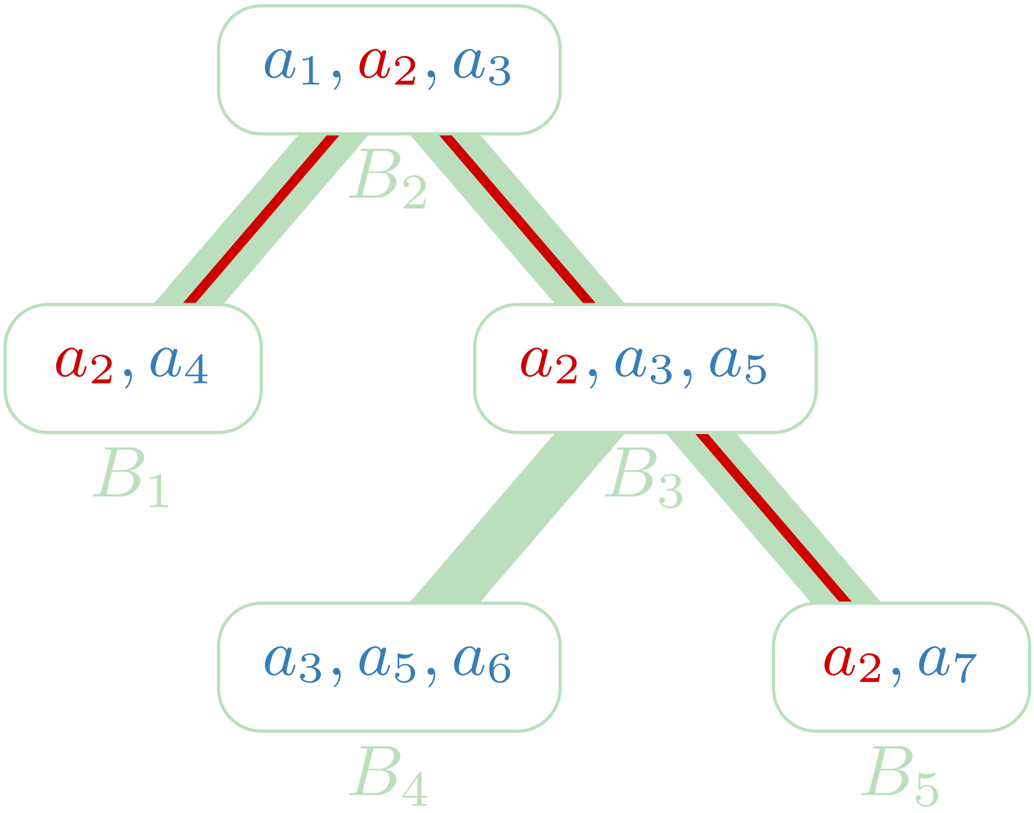
Tree Decompositions

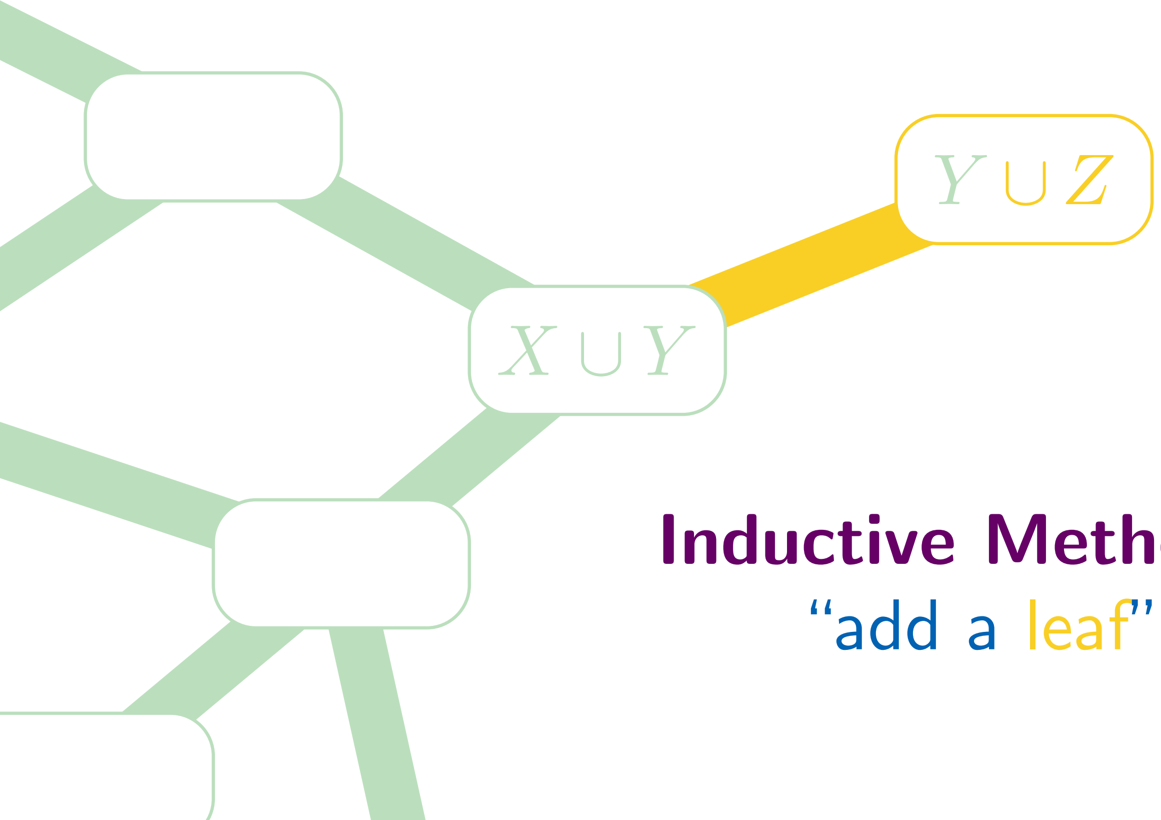
A tree decomposition of H are vertex sets (bags) $B_1, B_2, \ldots$ such that

- $B_1, B_2, \ldots$ are the vertices of a tree
- $v \in V(H) \Rightarrow \{B_i \mid v \in B_i\}$ subtree
- $uv \in E(H) \Rightarrow \exists i : u, v \in B_i$

The width is the maximum size of a bag -1.





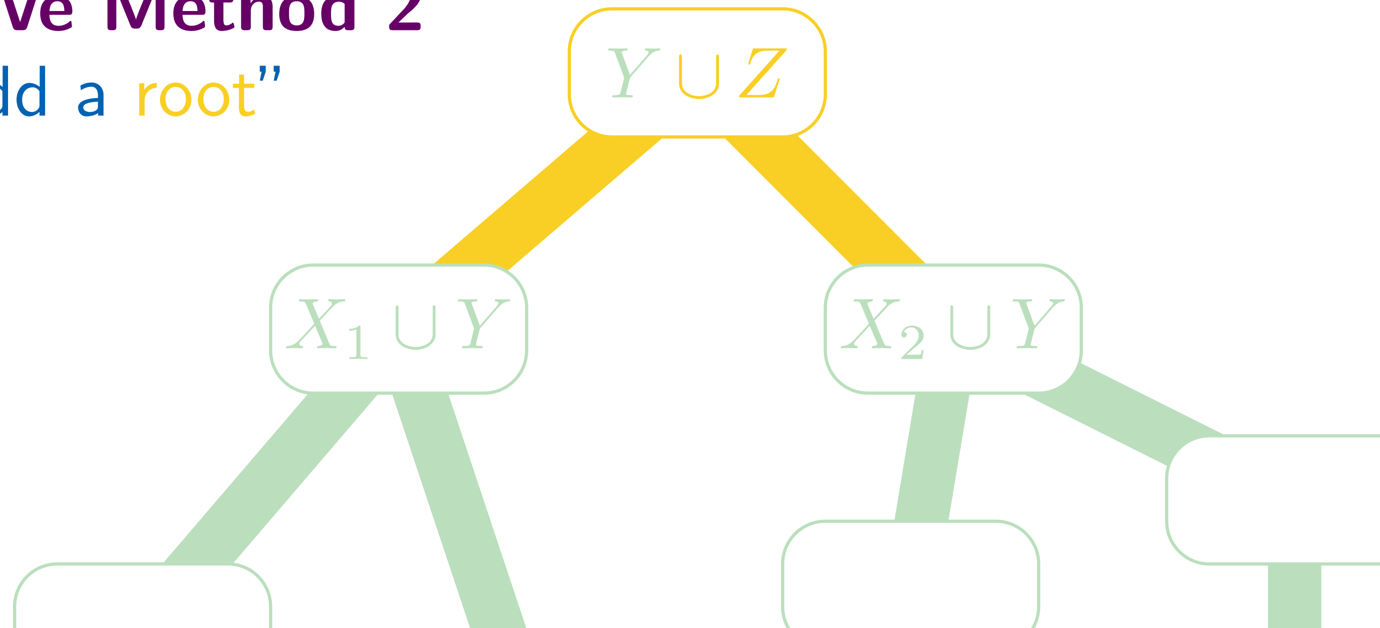


Inductive Method 1

"add a leaf"

Inductive Method 2

"add a root"



PRODUCT STRUCTURE THEORY

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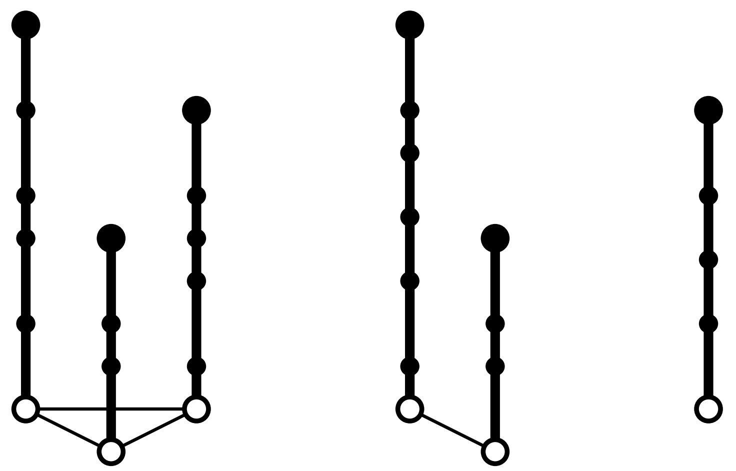
Tripod Partition Lemma

tripod
union of up to three vertical paths whose lower endpoints form a clique in G



tripods with 3,2,1 legs

Tripod Partition Lemma

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union of up to three vertical paths whose lower endpoints form a clique in G

Let ▷ G^+ planar triangulation, T BFS tree rooted at an outer vertex

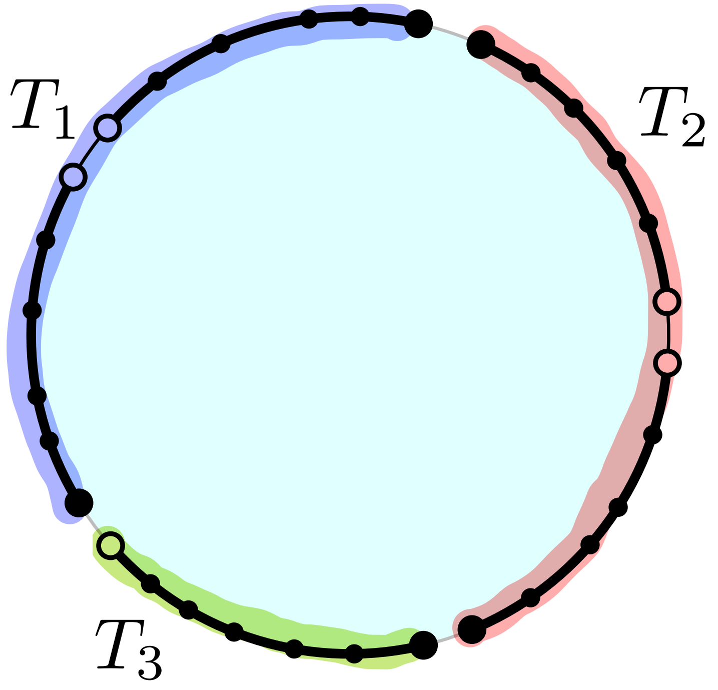
▷ T_1, T_2, T_3 pairwise disjoint tripods

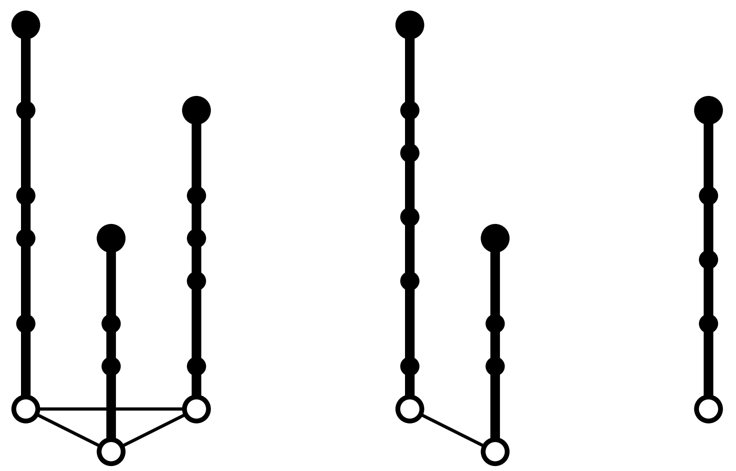
▷ $F = [T_1, T_2, T_3]$ cycle

▷ G near-triangulation on all vertices on and inside F

Then ▷ there exists a partition $\mathcal{T}$ of G into tripods with $T_1, T_2, T_3 \in \mathcal{T}$

▷ $H = G/\mathcal{T}$ has tree-decomposition of width 3 with a bag containing T_1, T_2, T_3





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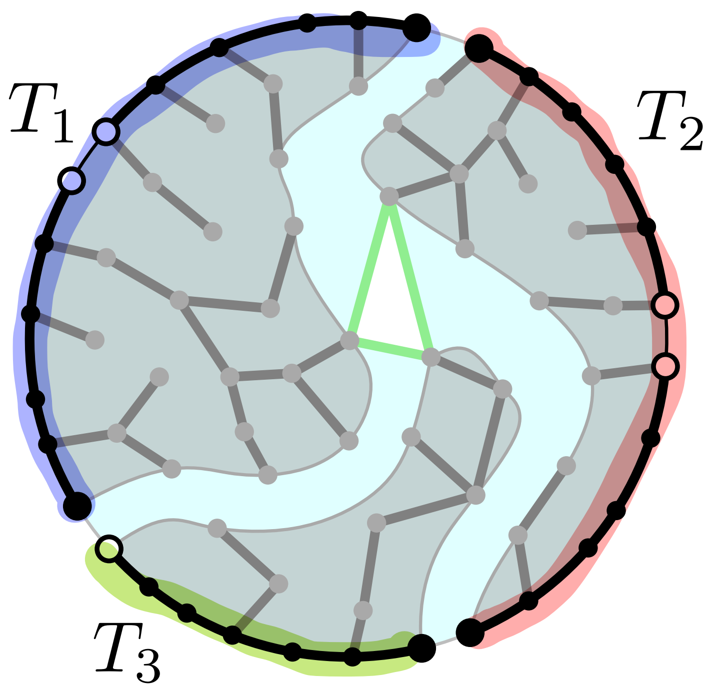

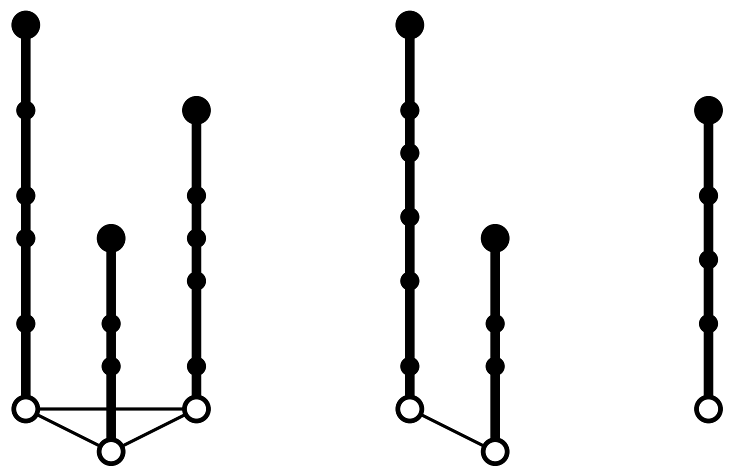

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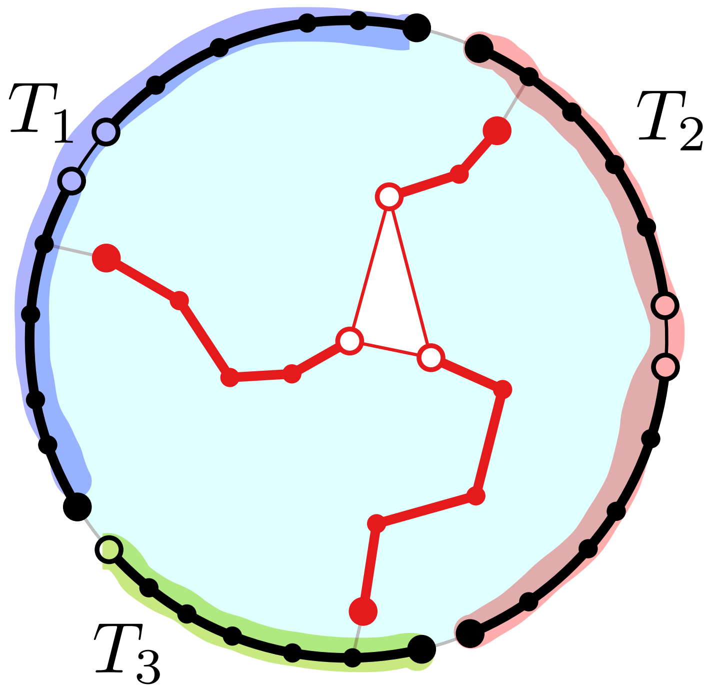

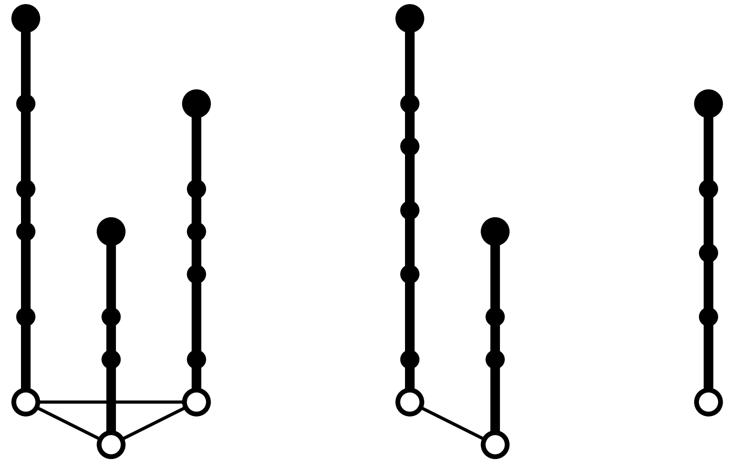

tripods with 3,2,1 legs

Tripod Partition Lemma

tripod
union of up to three vertical paths whose lower endpoints form a clique in G

Let ▷ G^+ planar triangulation, T BFS tree rooted at an outer vertex

▷ T_1, T_2, T_3 pairwise disjoint tripods

▷ $F = [T_1, T_2, T_3]$ cycle

▷ G near-triangulation on all vertices on and inside F

Then ▷ there exists a partition $\mathcal{T}$ of G into tripods with $T_1, T_2, T_3 \in \mathcal{T}$

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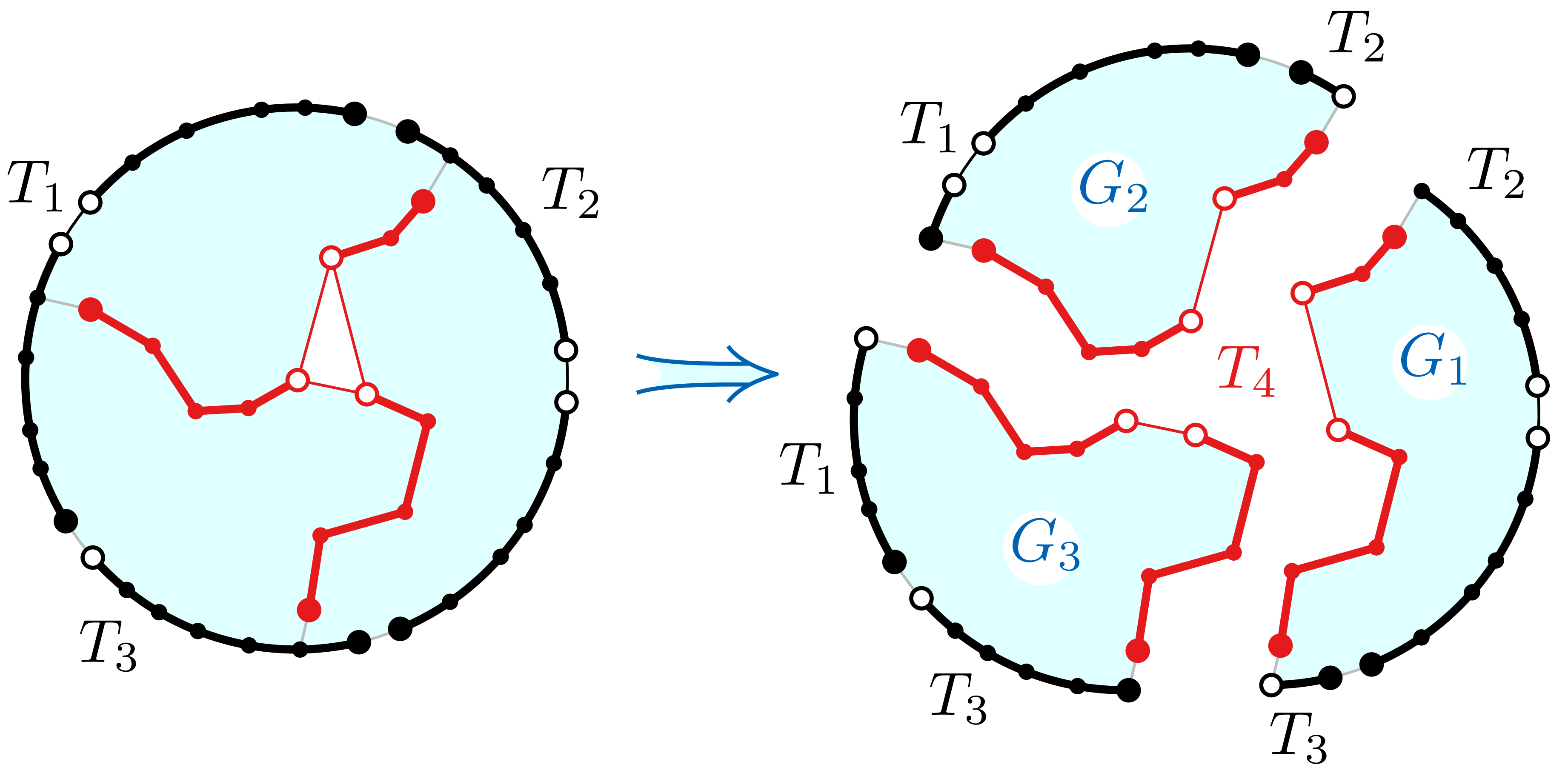

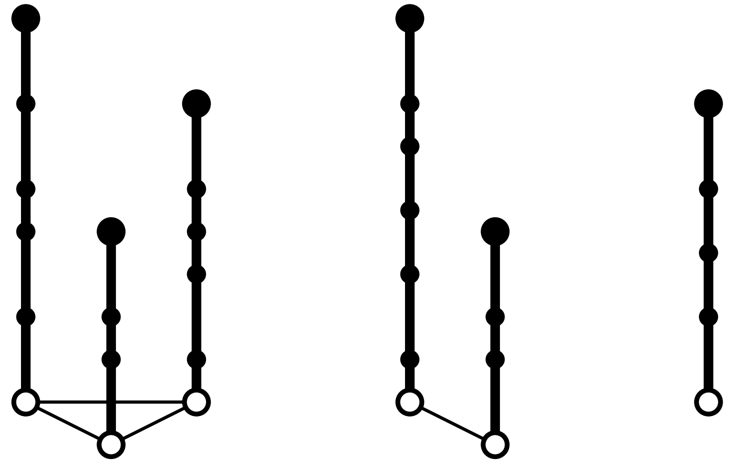

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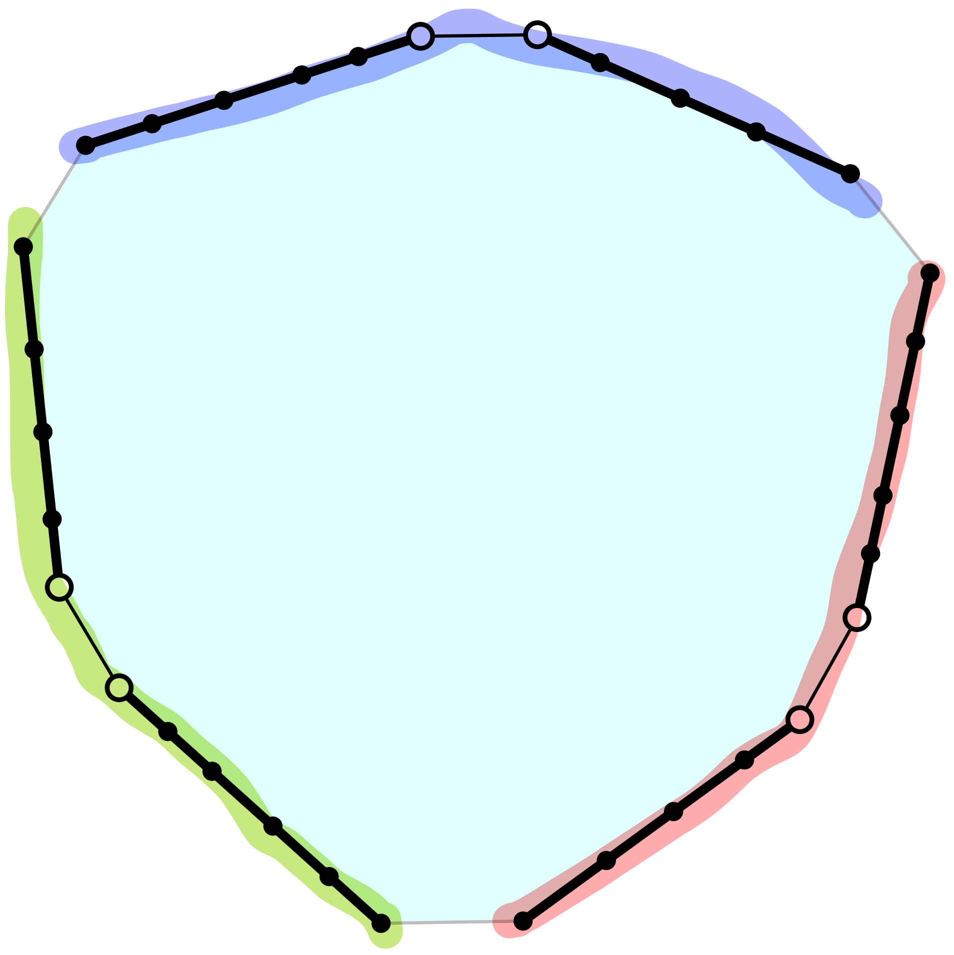

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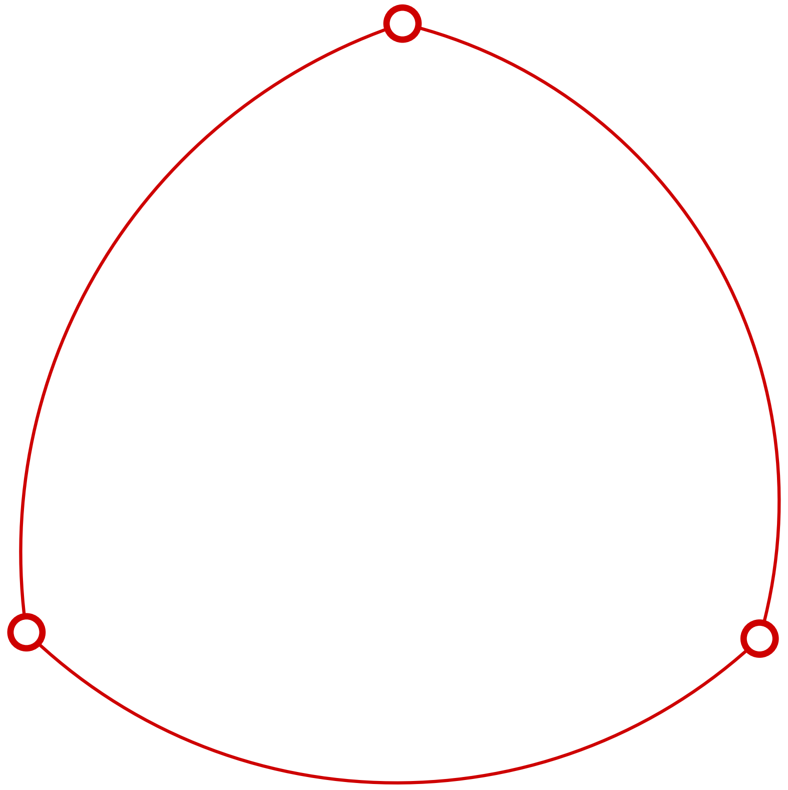

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- G is a planar triangulation
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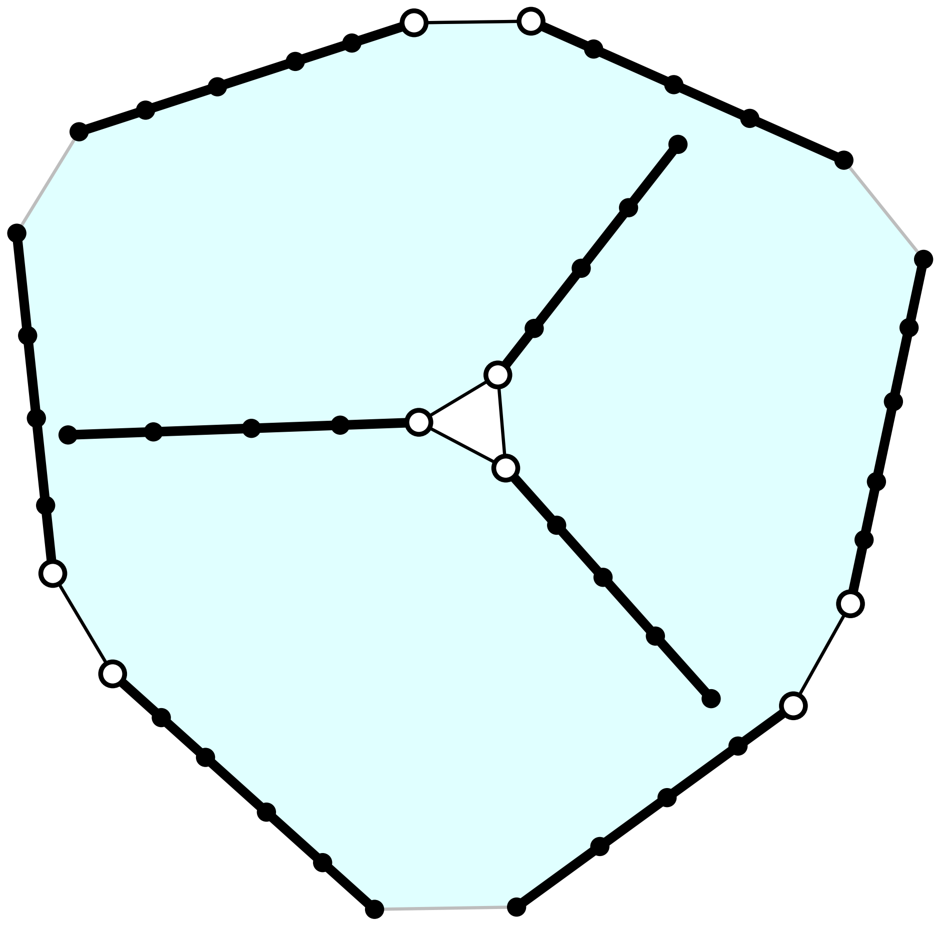

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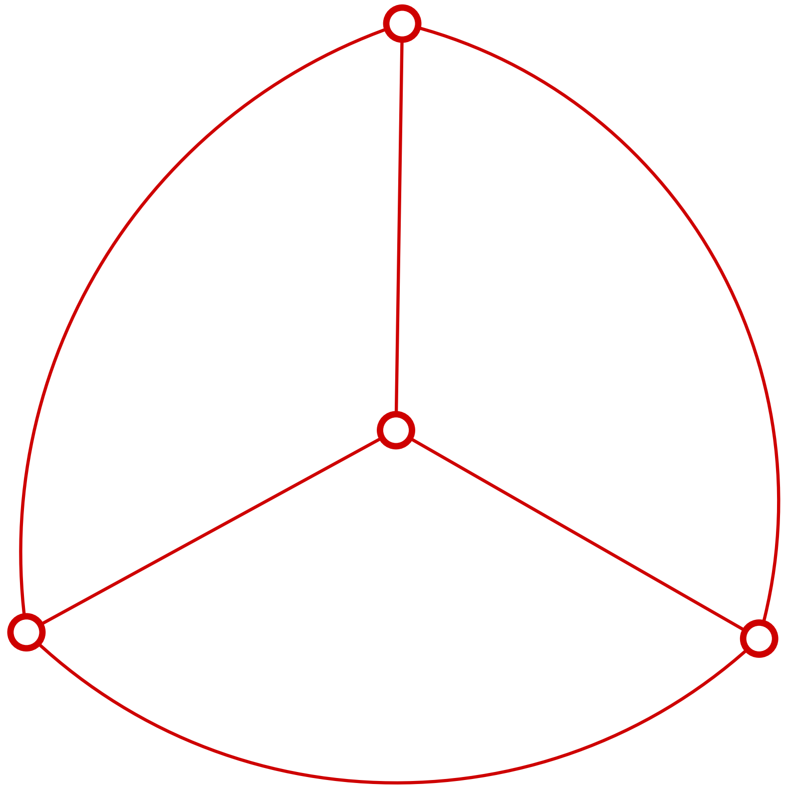

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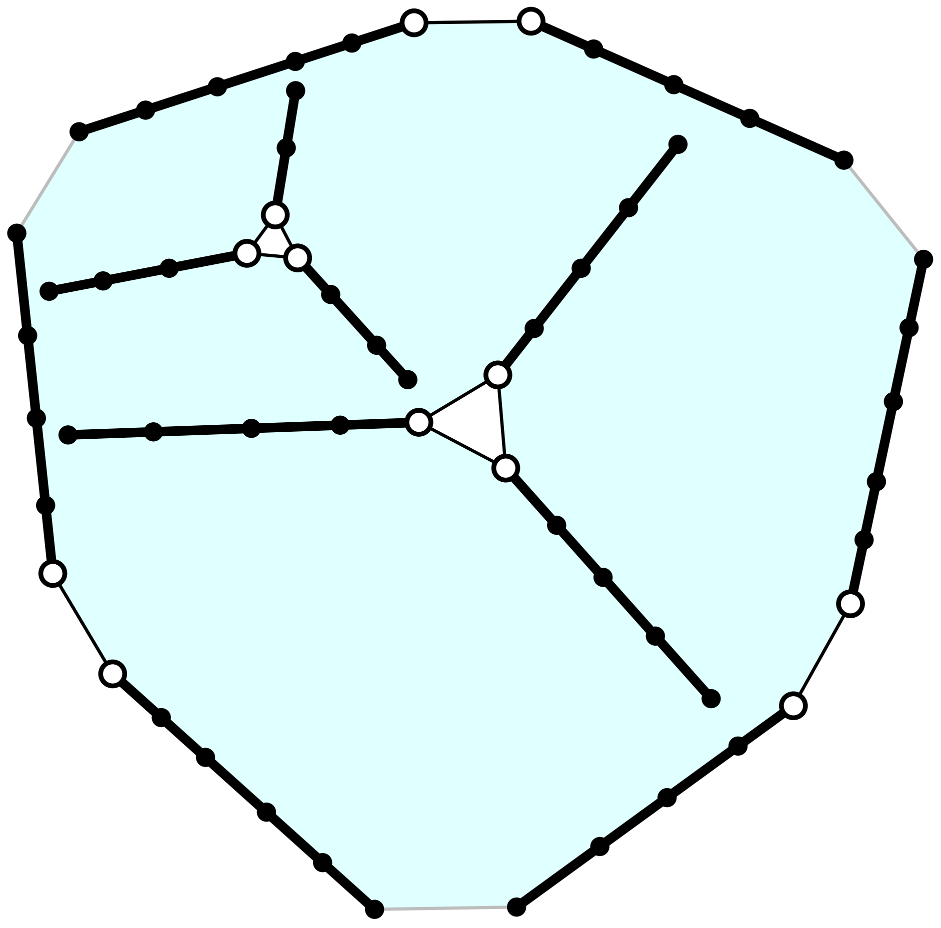

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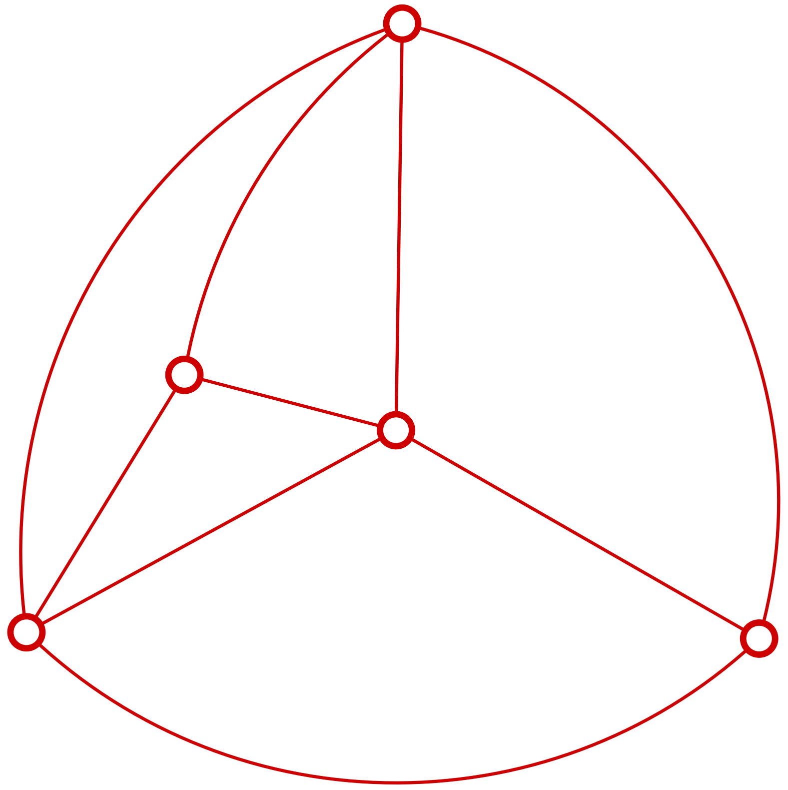

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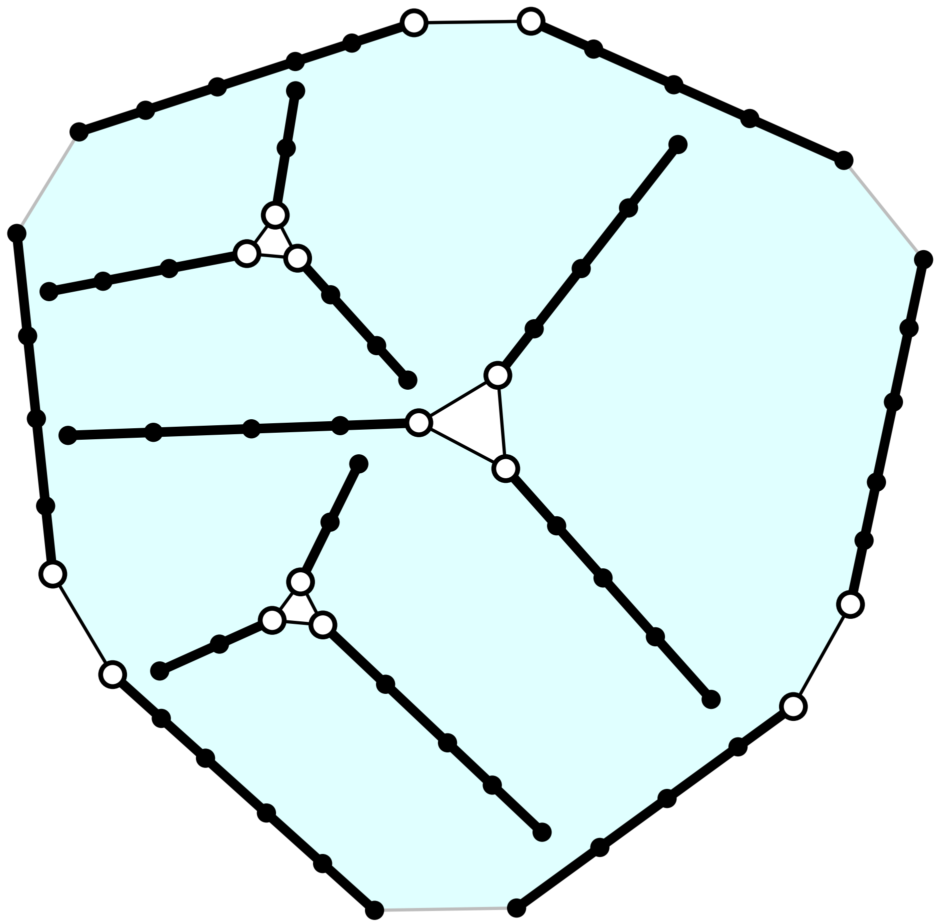

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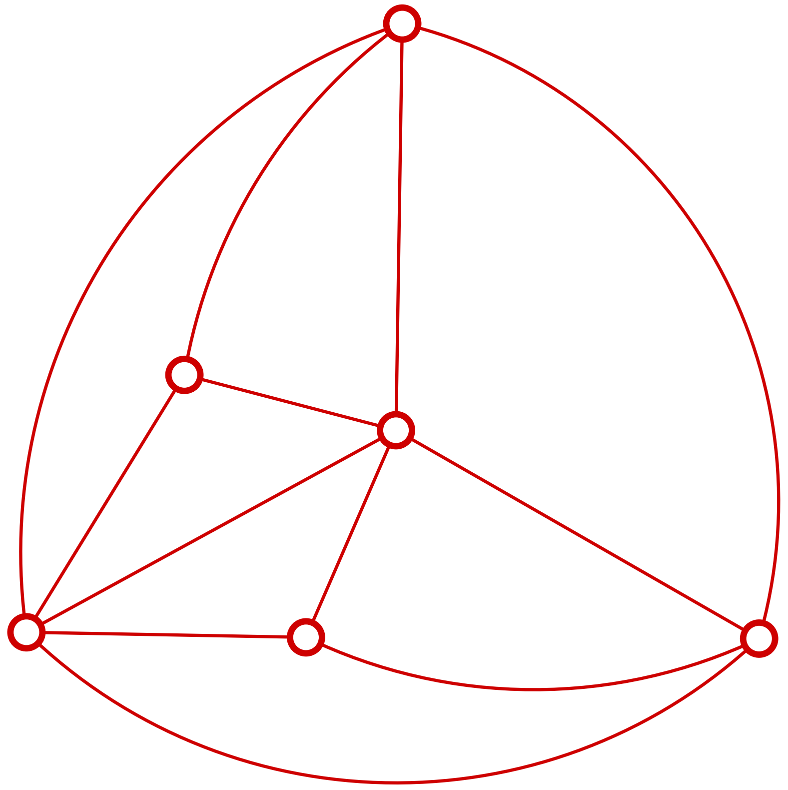

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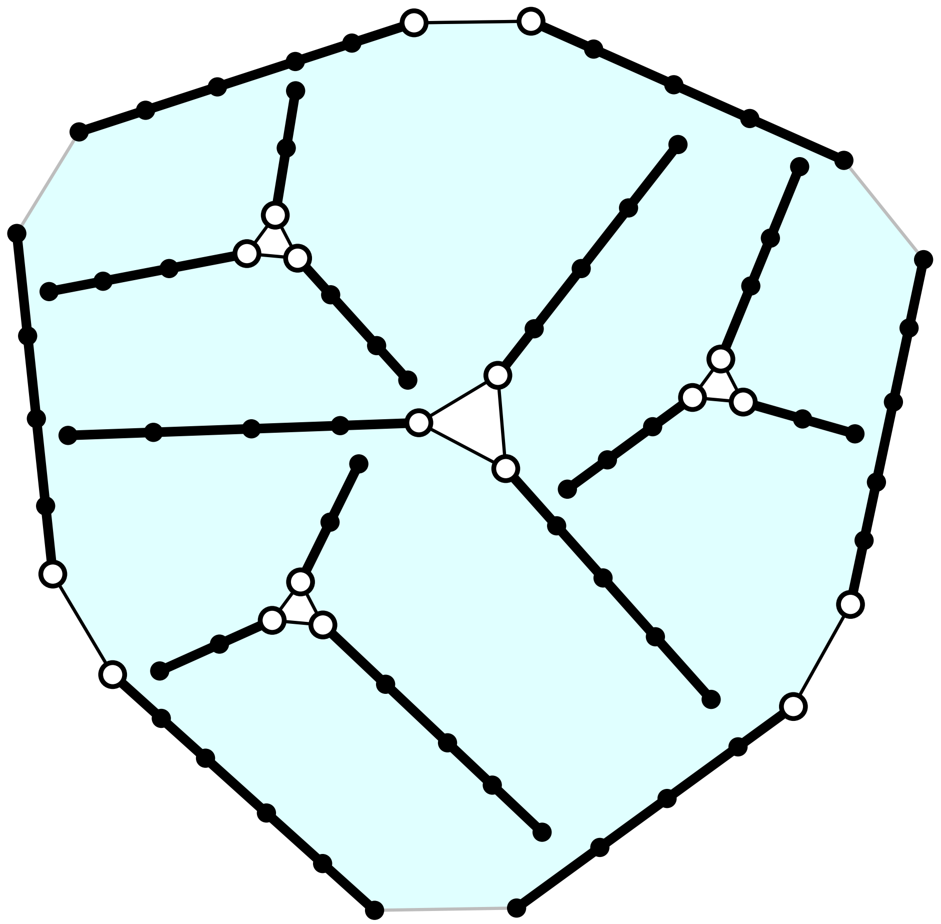

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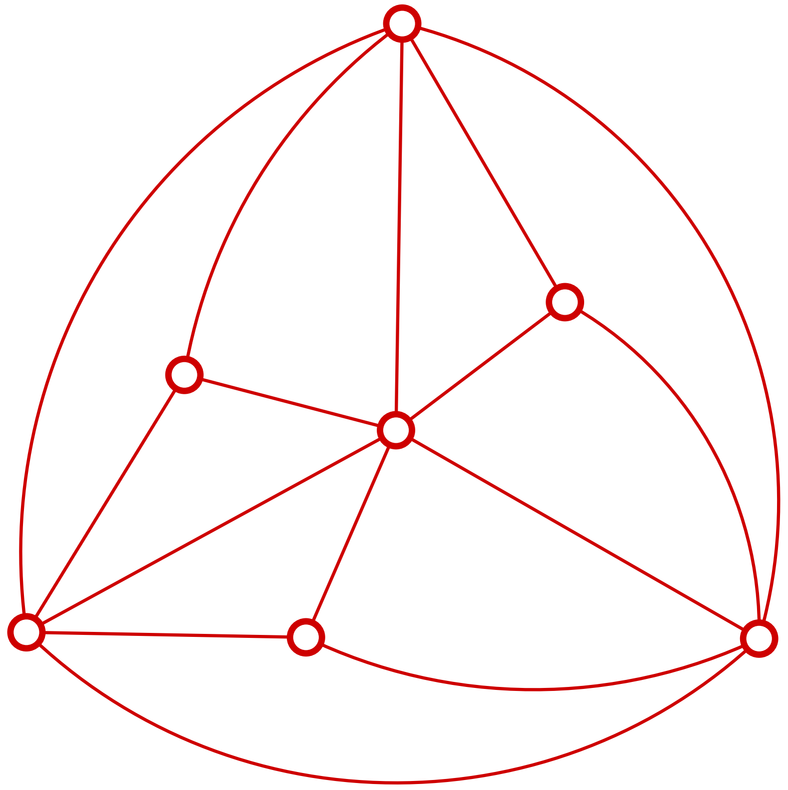

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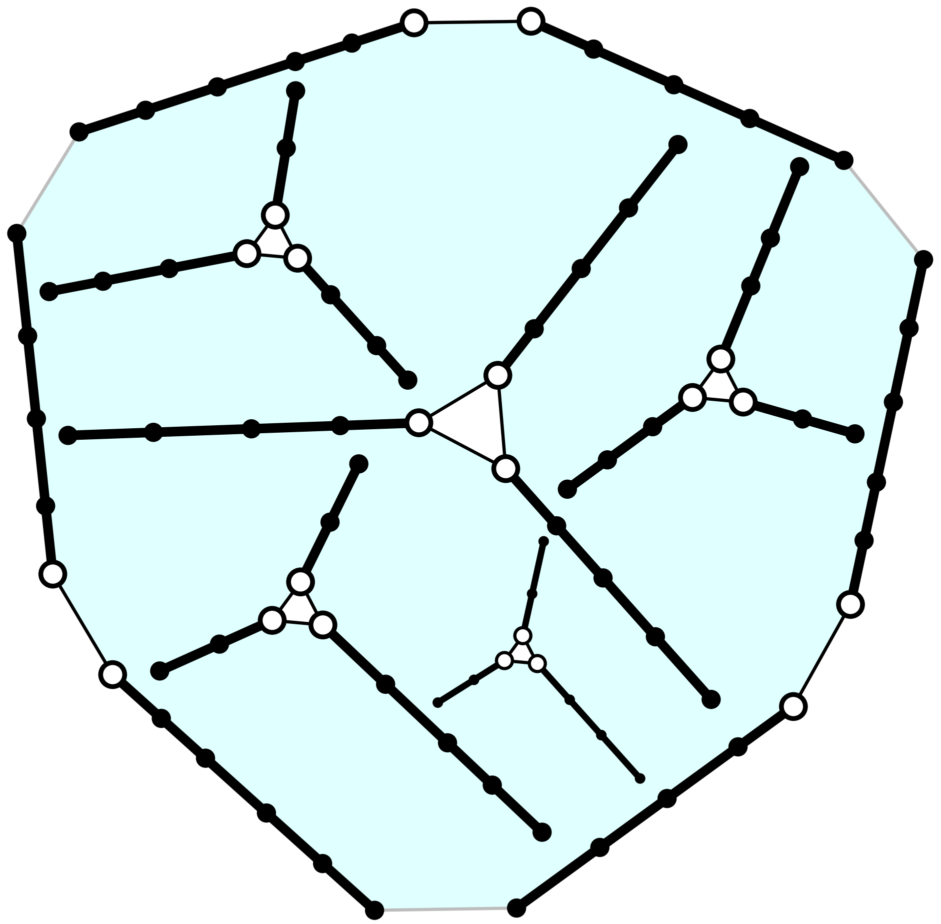

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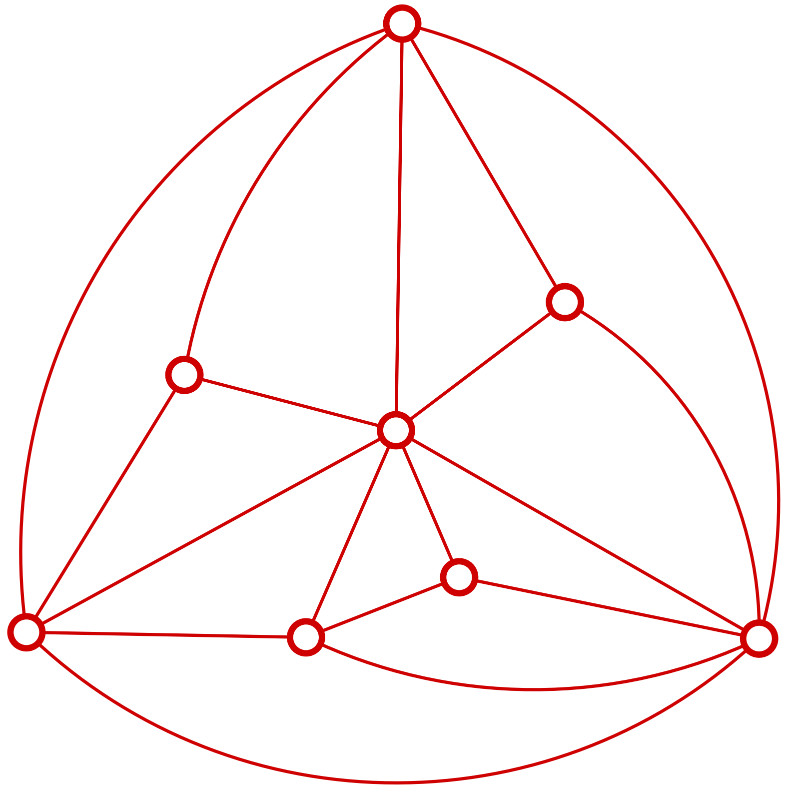

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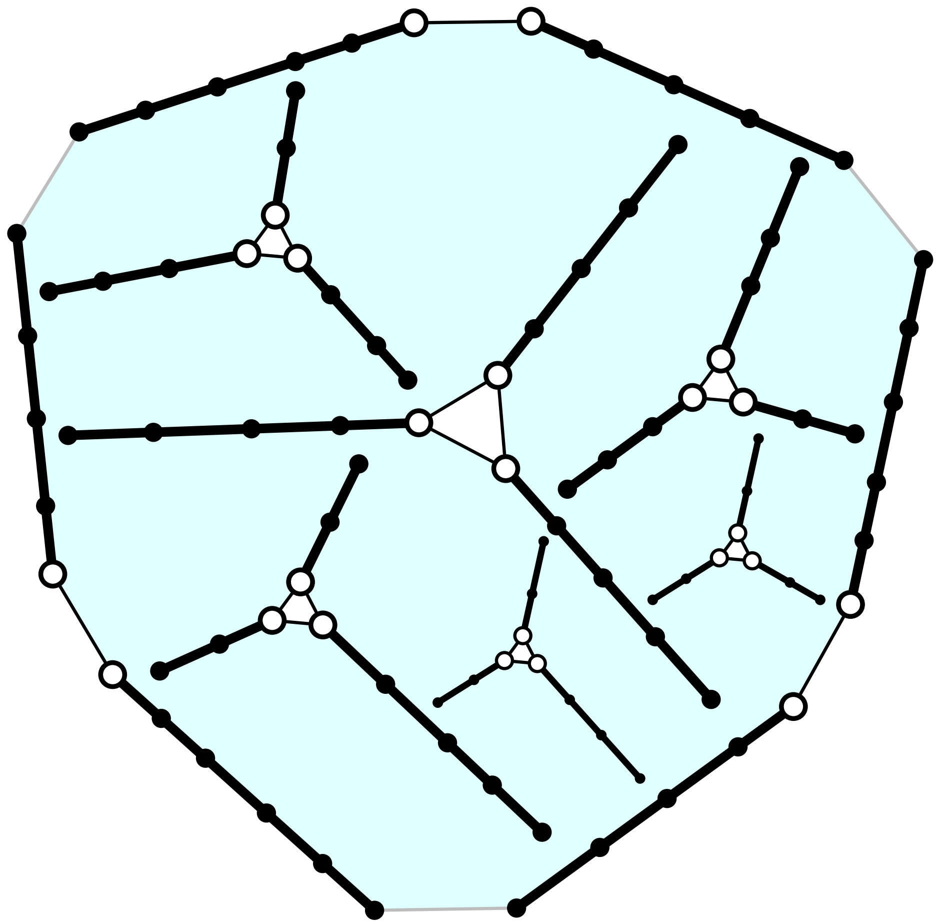

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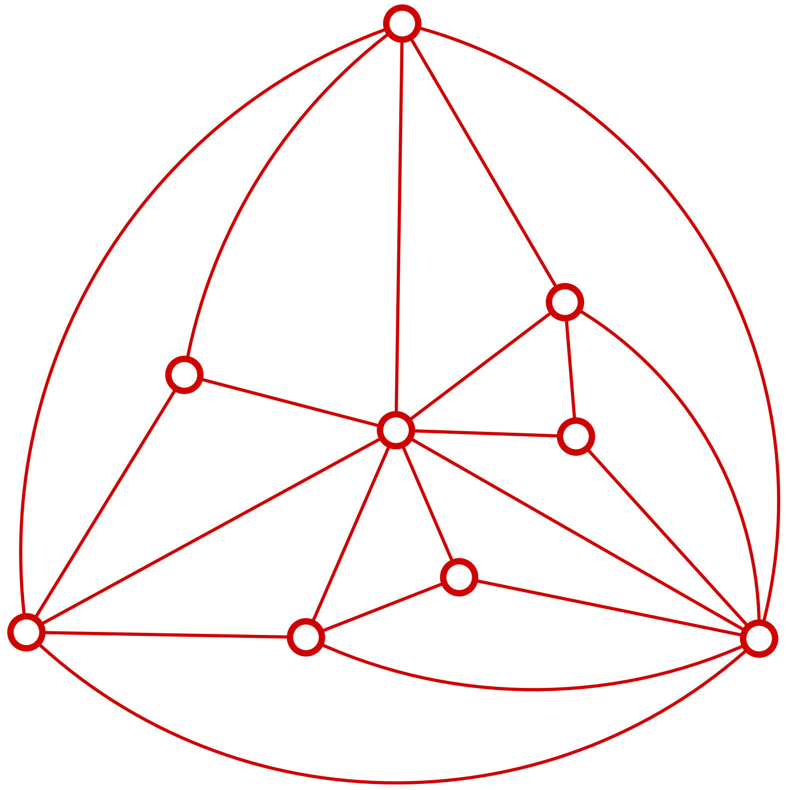

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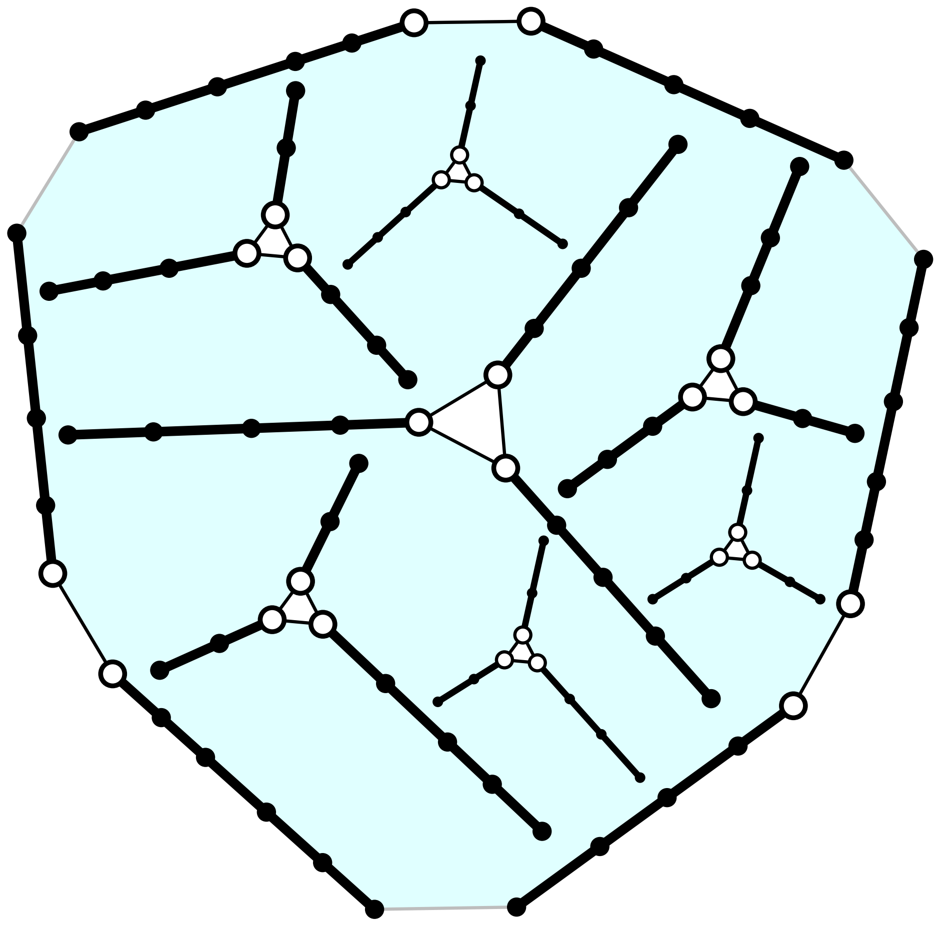

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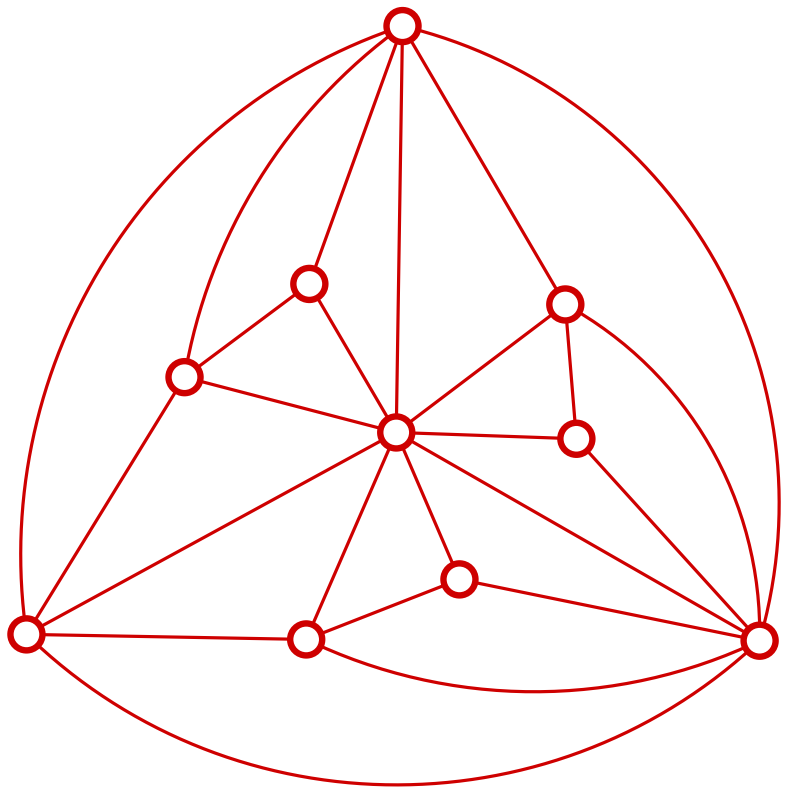

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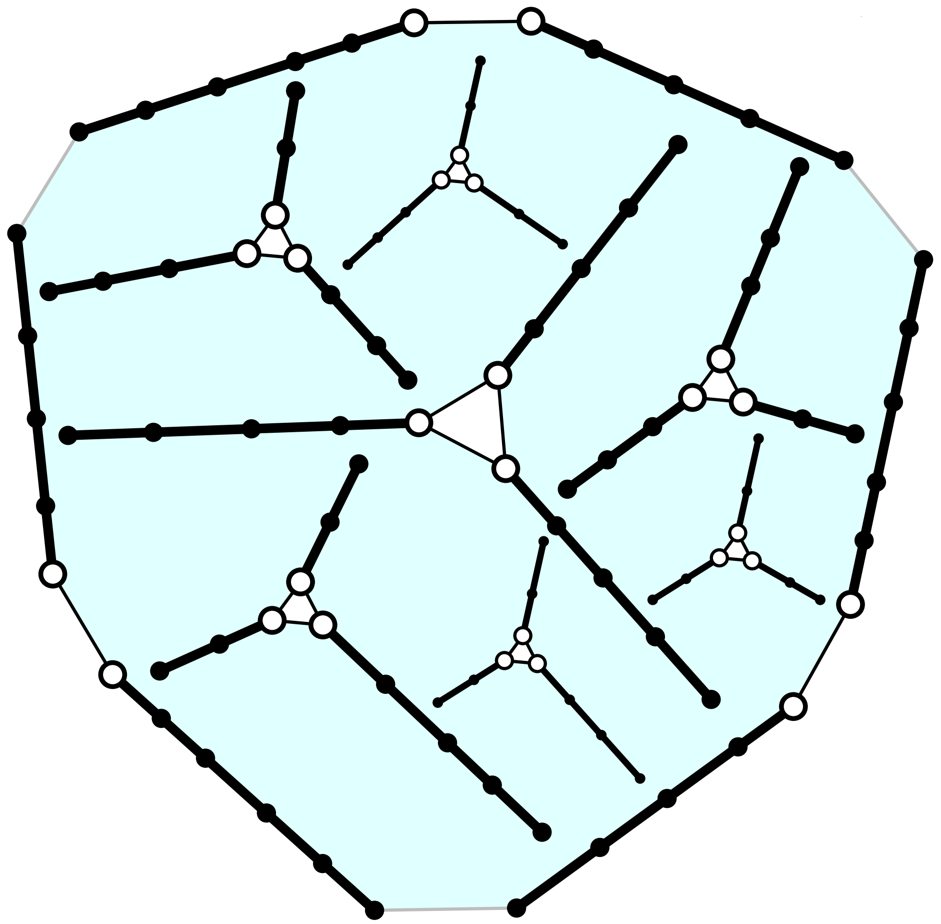
TRIPODS SUMMARY



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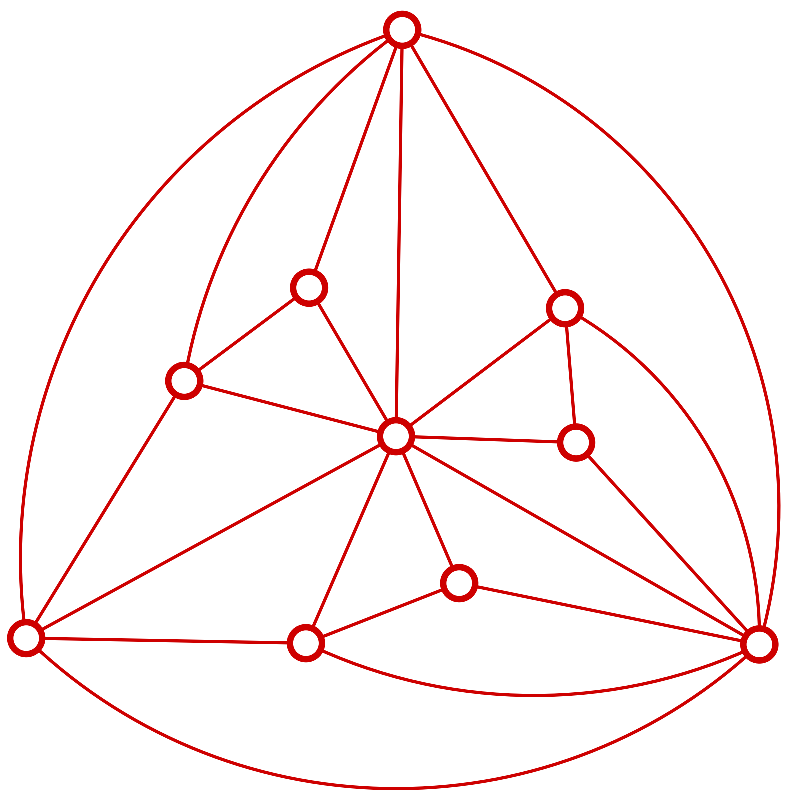

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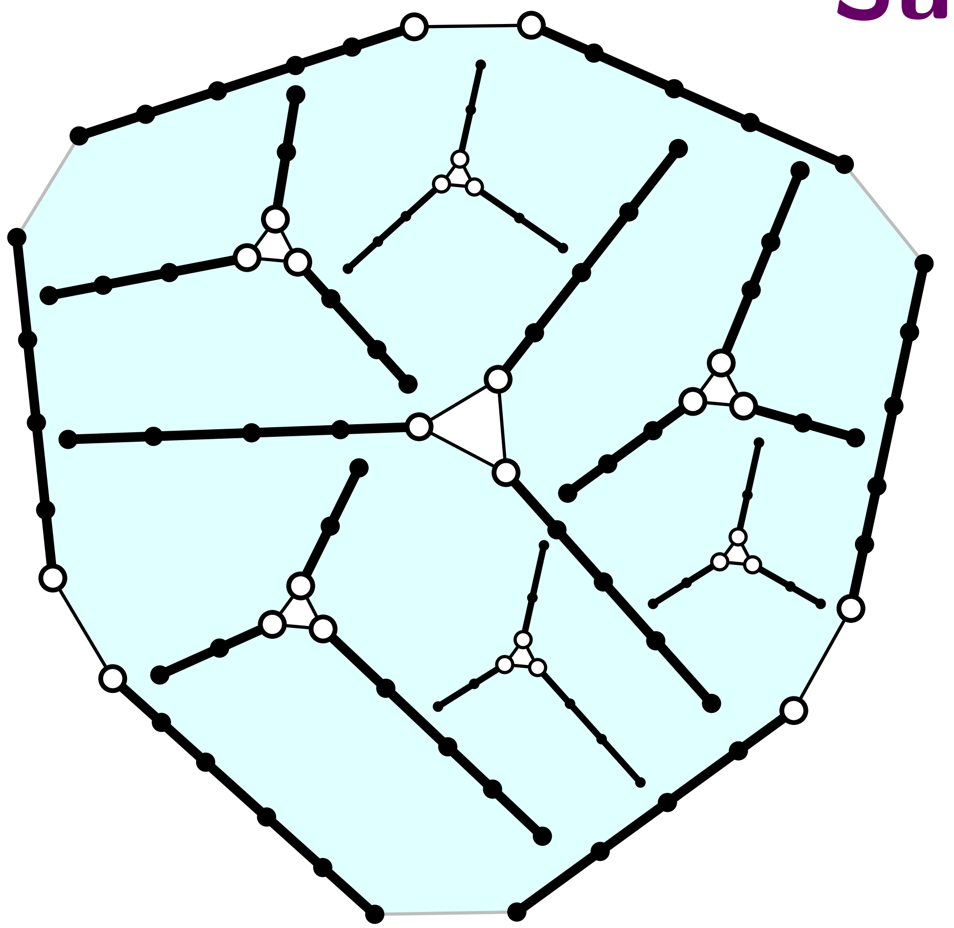
Planar Product Structure Theorems [DJMMUW '19]

Any **planar** graph G is a subgraph of the product $H \boxtimes P \boxtimes K_3$ of K_3, a path P, and a graph H a planar 3-tree

Summary : vertical paths & tripods



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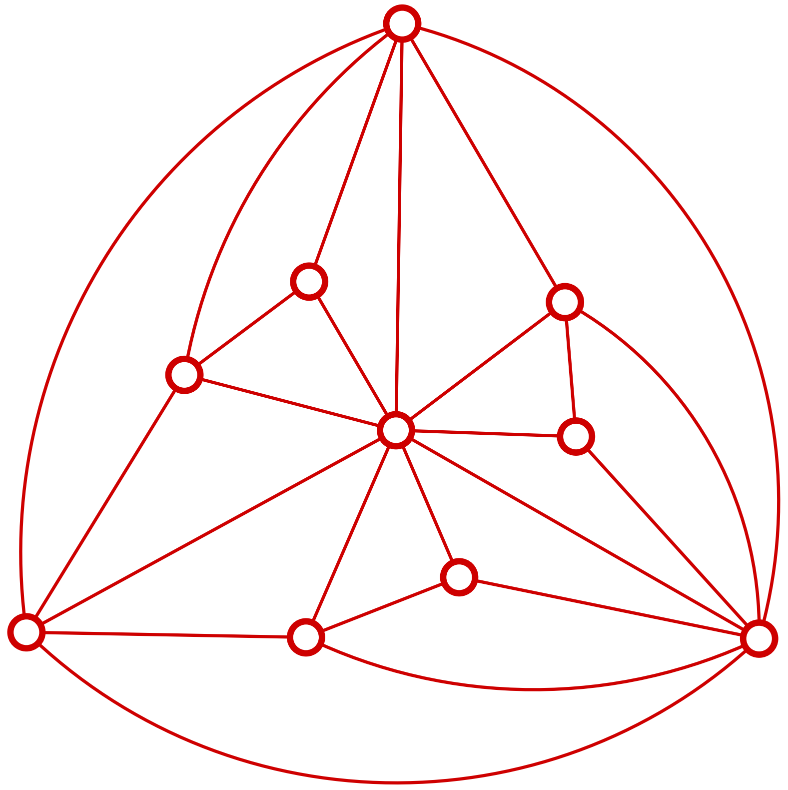

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Any **planar** graph G is a subgraph of the product $H \boxtimes P$ of a path P and a graph H of simple treewidth 8

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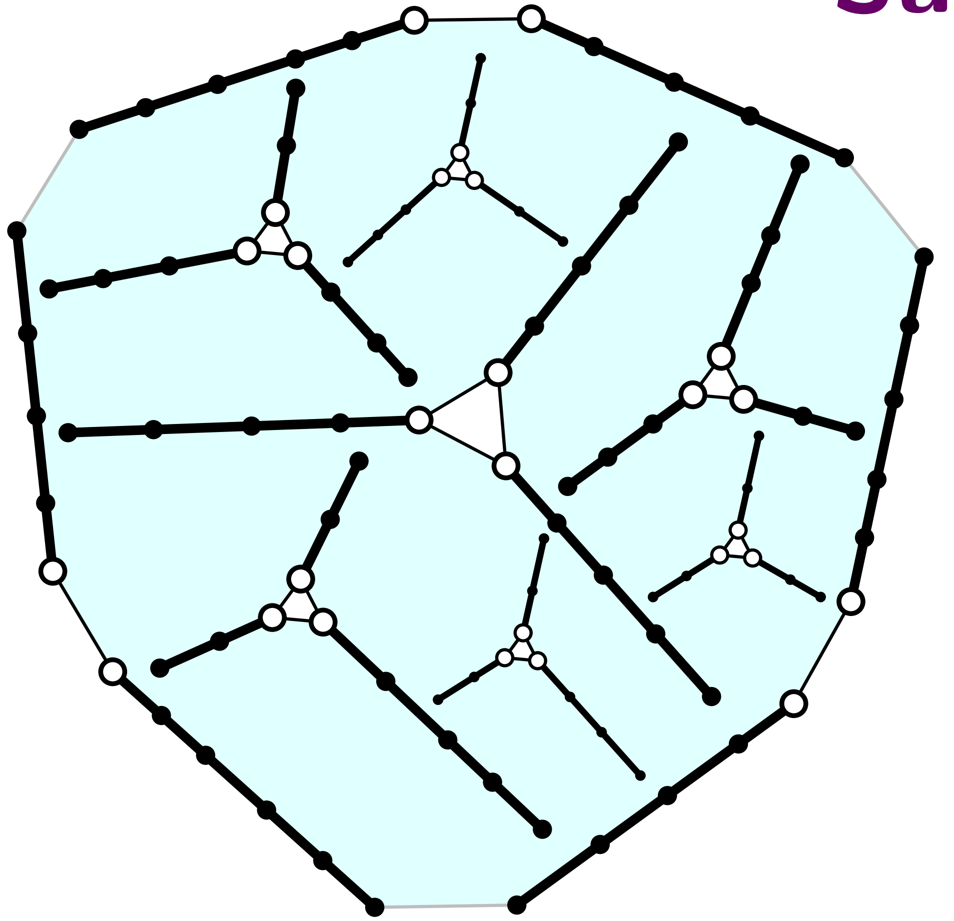

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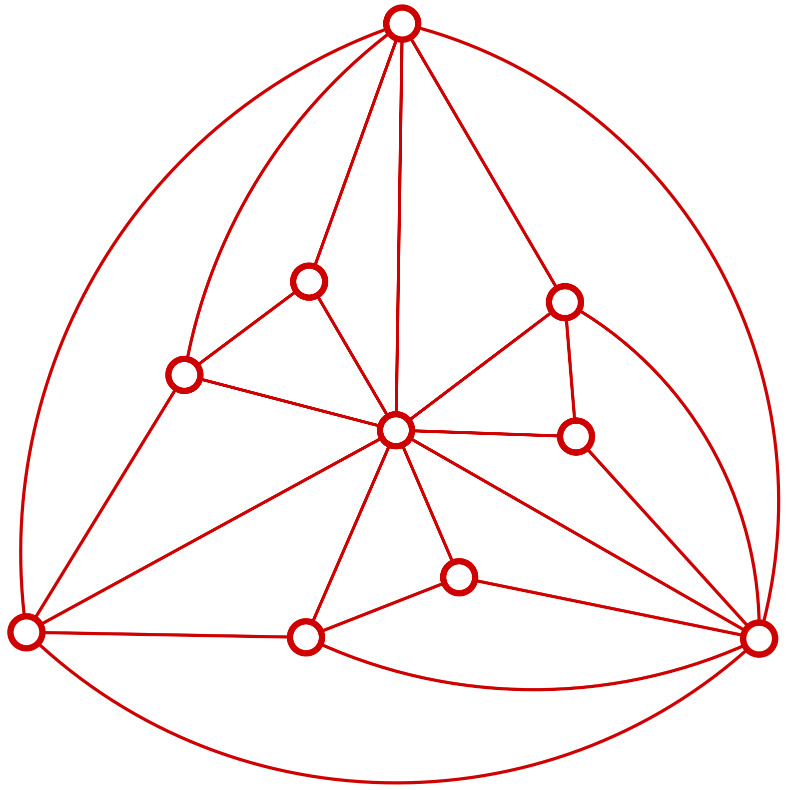

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Planar Product Structure Theorems [DJMMUW '19, UWY '21+].

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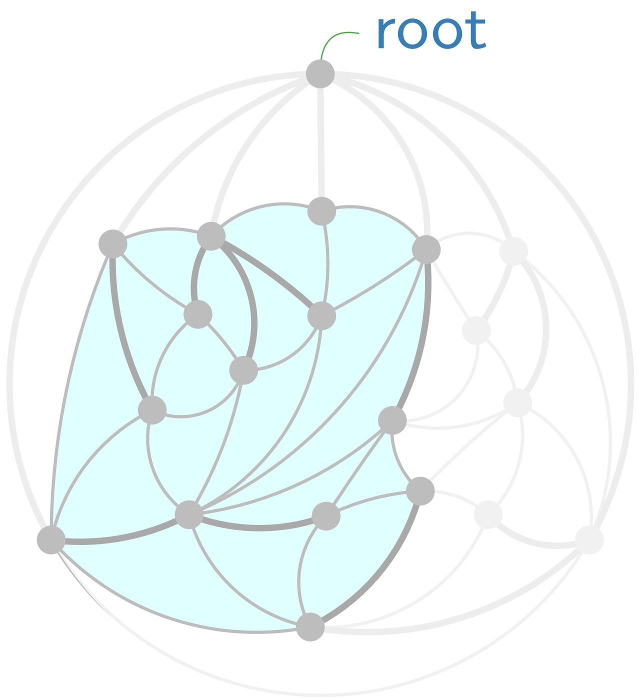
The Main Lemma



Let ▷ G^+ planar triangulation, T BFS tree rooted at an outer vertex

▷ $P_1, \dots, P_k$ pairwise disjoint vertical paths, $k \leq 6$

▷ $F = [P_1, \dots, P_k]$ cycle

▷ G near-triangulation on all vertices on and inside F

Then ▷ there exists a partition $\mathcal{P}$ of G into vertical paths with $P_1, \dots, P_k \in \mathcal{P}$

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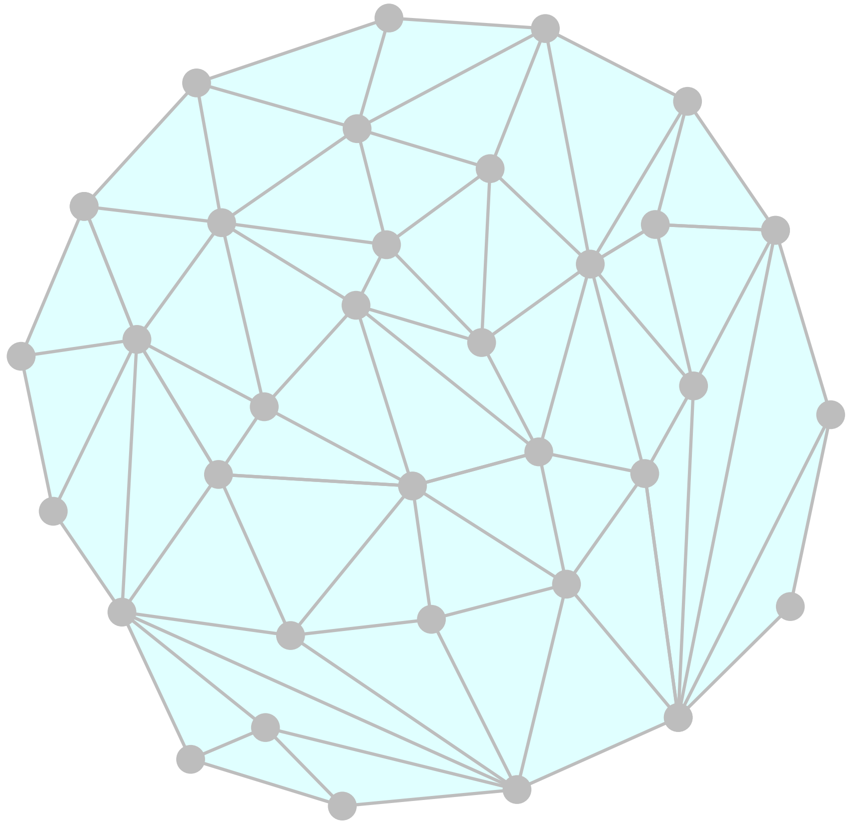

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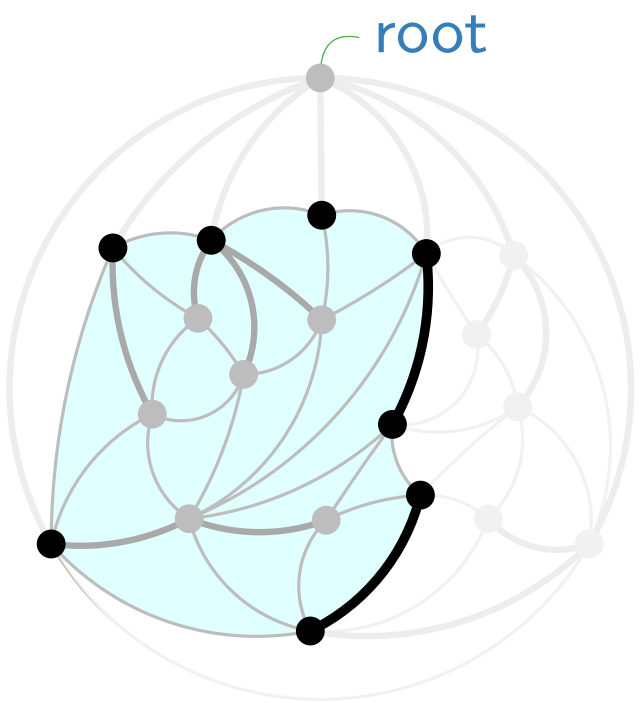

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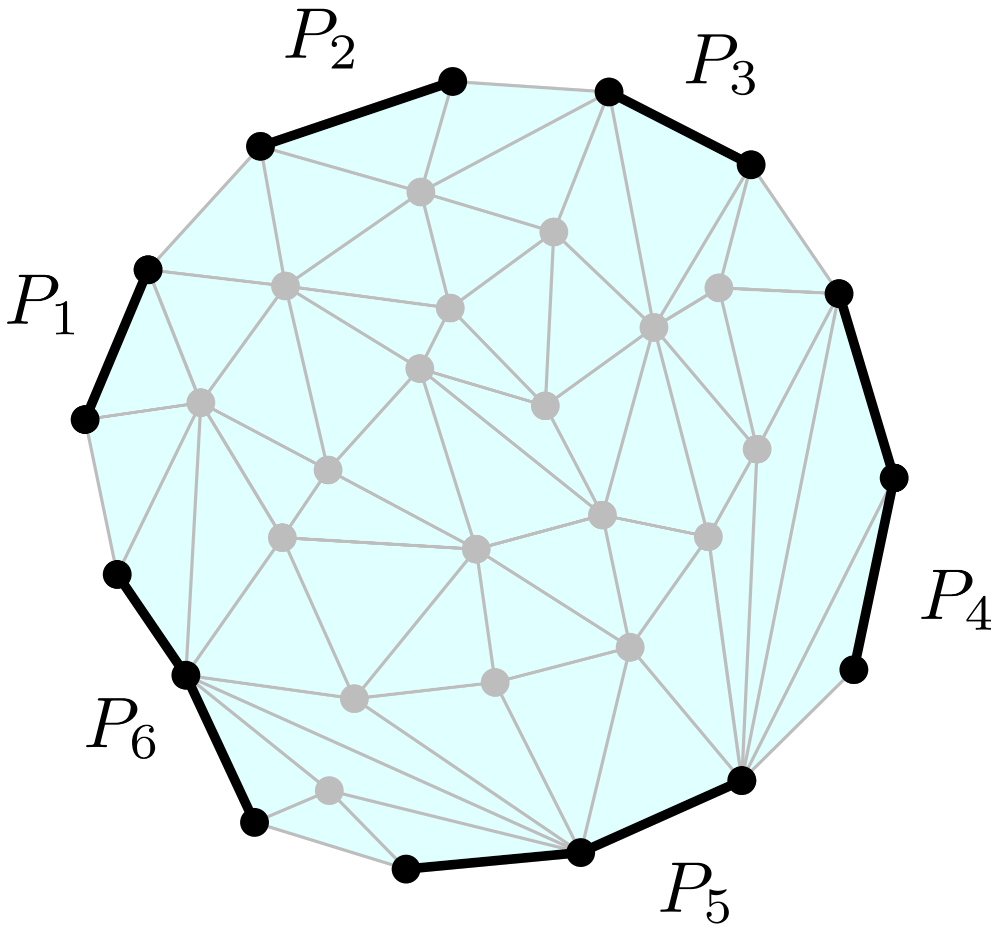

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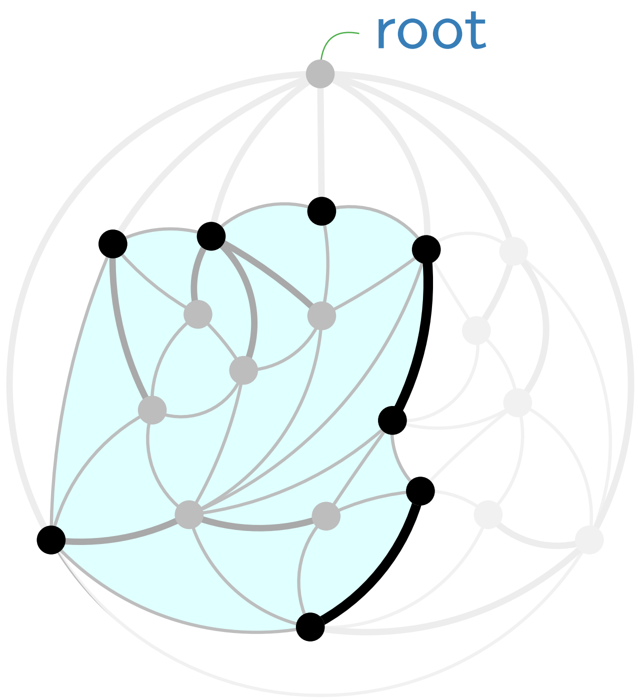

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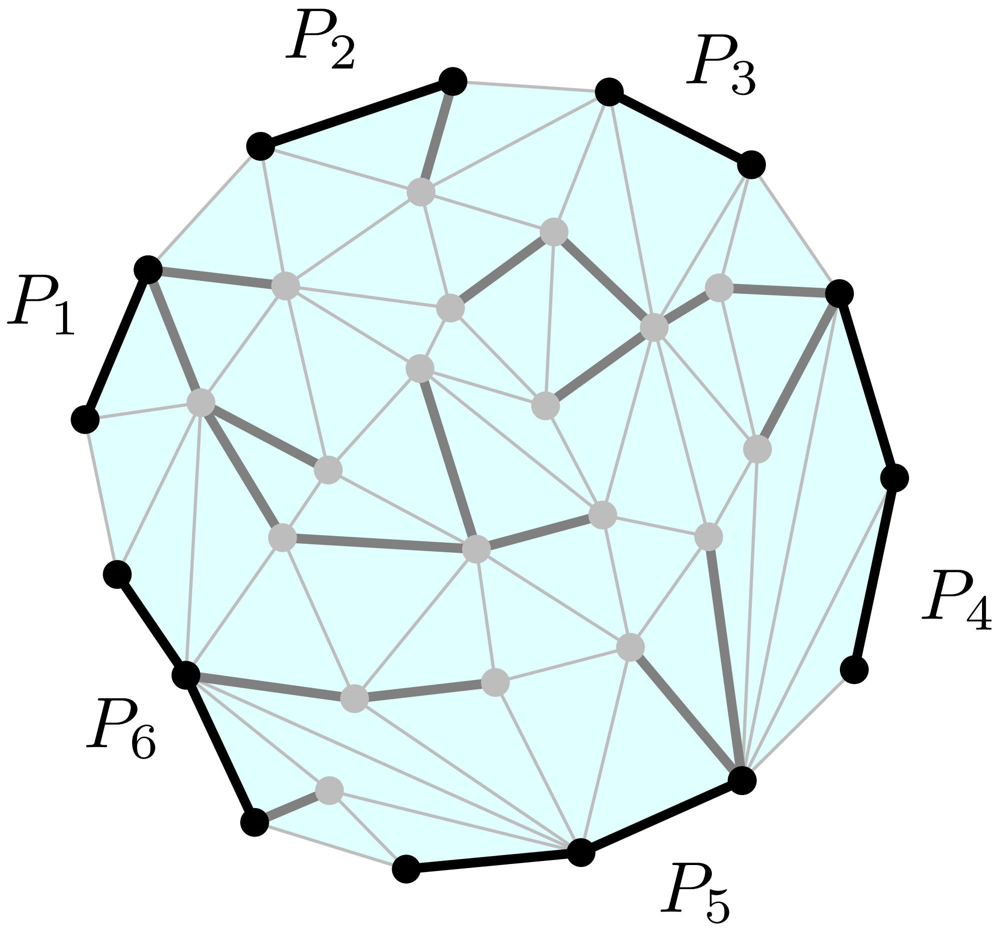
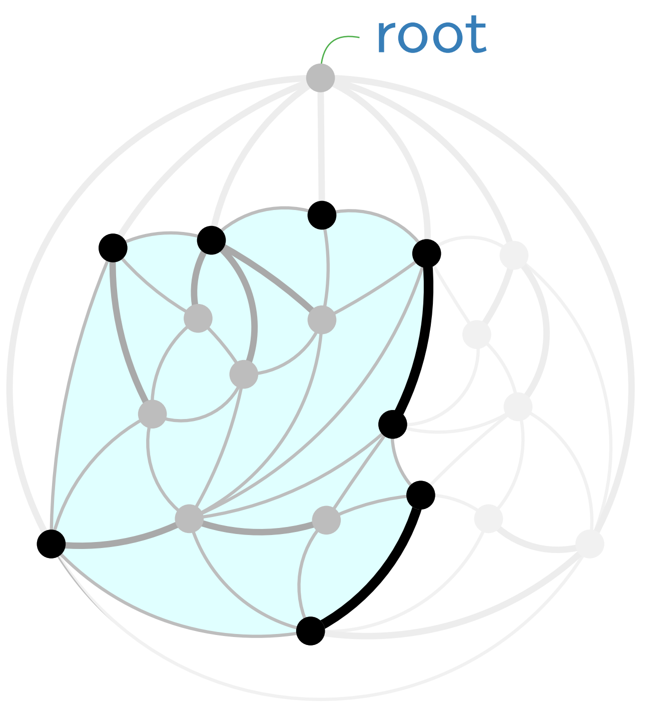


Proof of the **Main Lemma**.

▷ consider the parts of T in G

The Main Lemma



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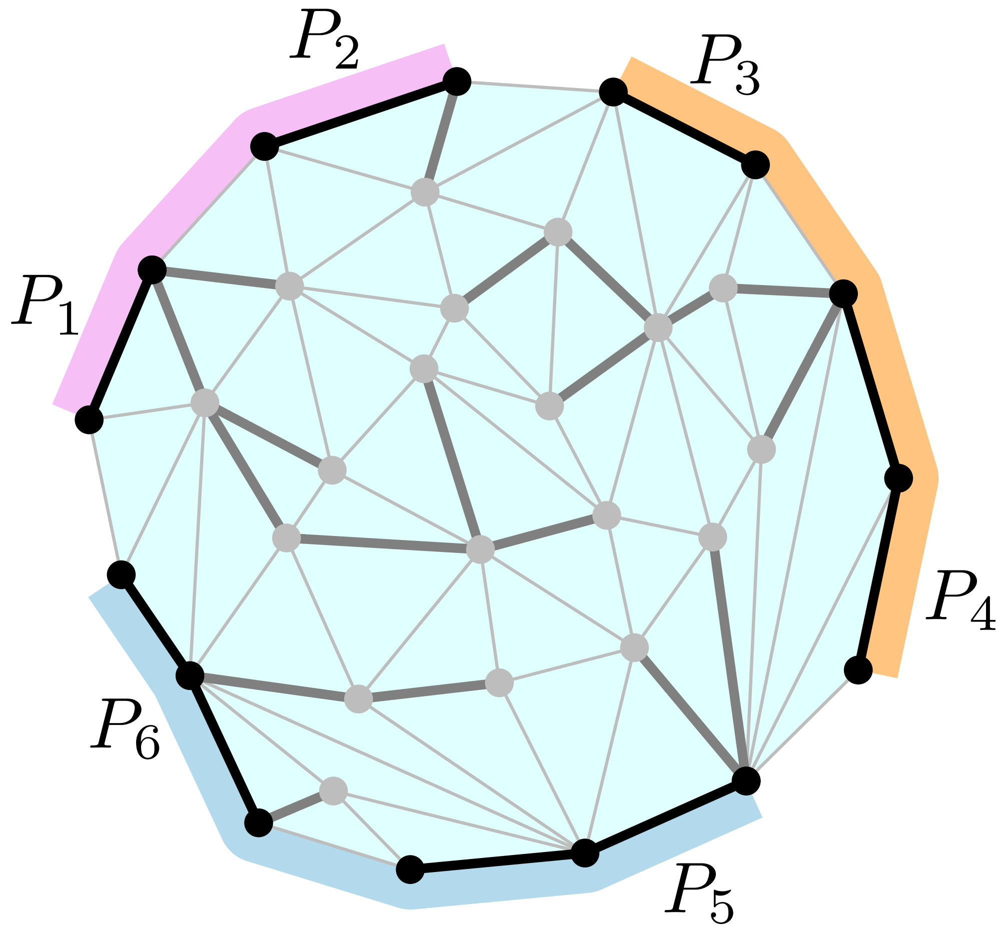
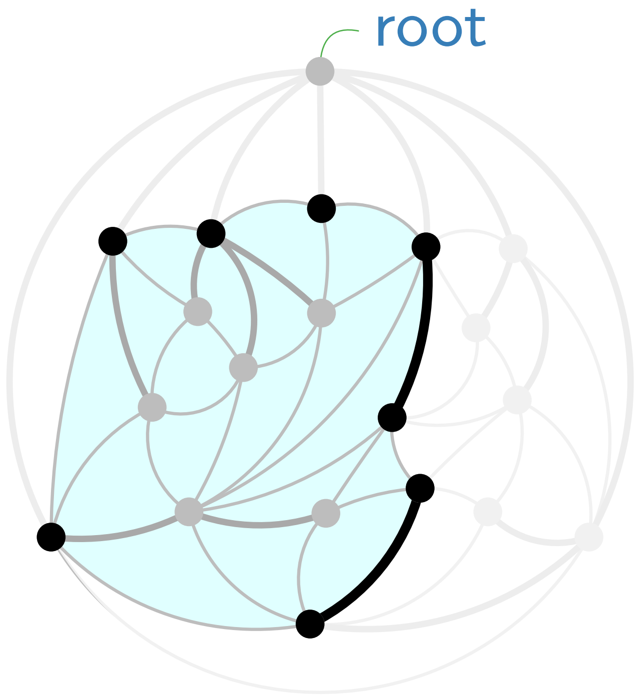


Proof of the **Main Lemma**.

▷ consider the parts of T in G

▷ exactly 3 groups of at most 2 paths each

The Main Lemma



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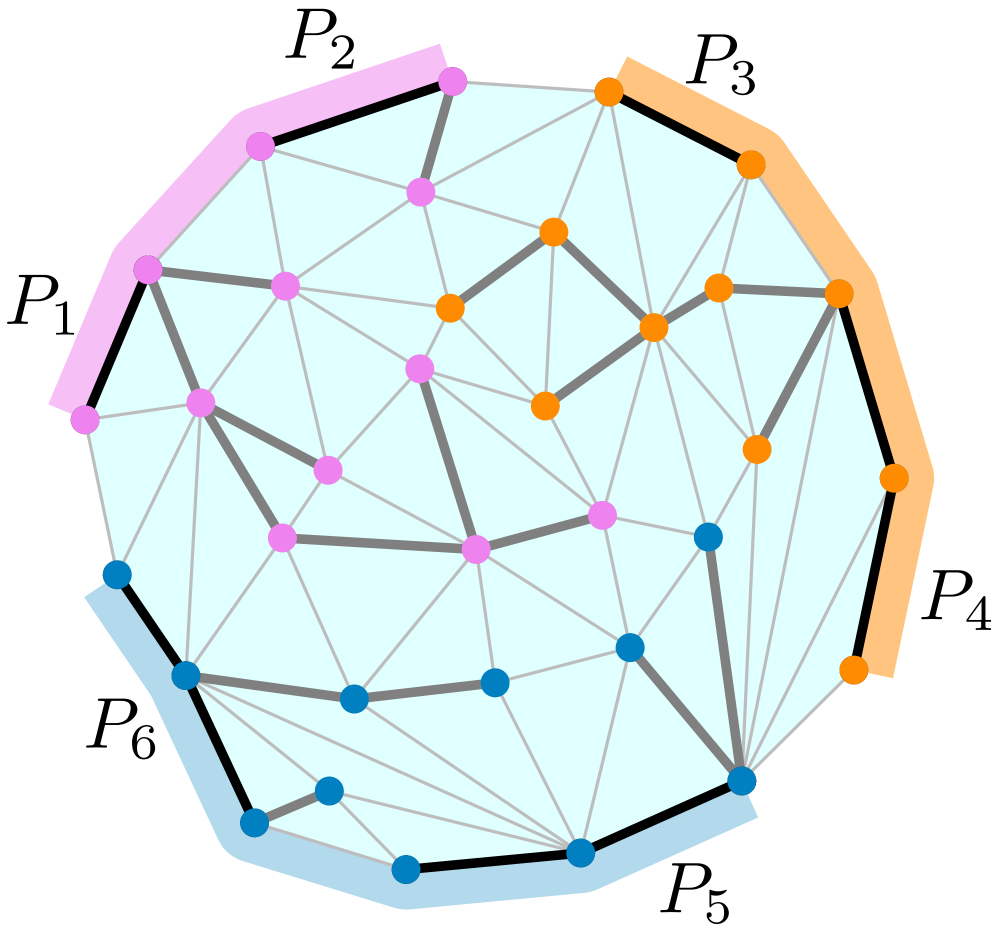


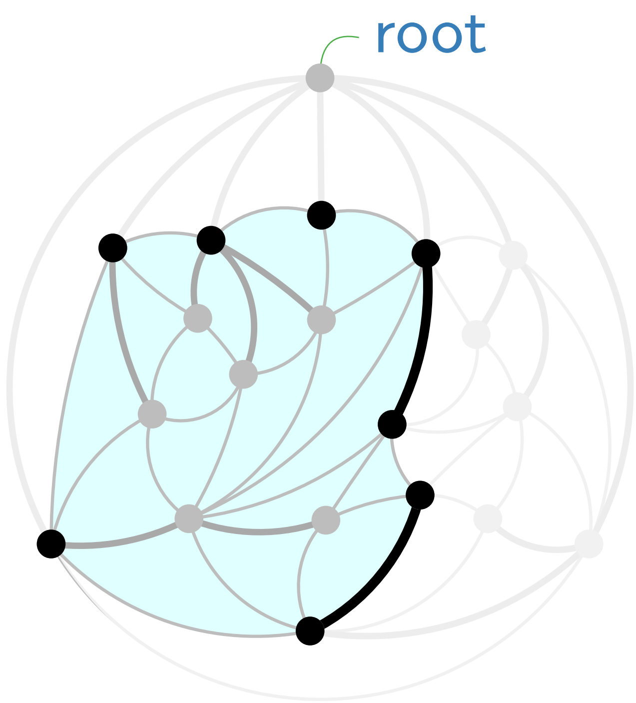
Proof of the **Main Lemma**.

▷ consider the parts of T in G

▷ exactly 3 groups of at most 2 paths each

▷ 3-coloring of $V(G)$ by going along T

The Main Lemma



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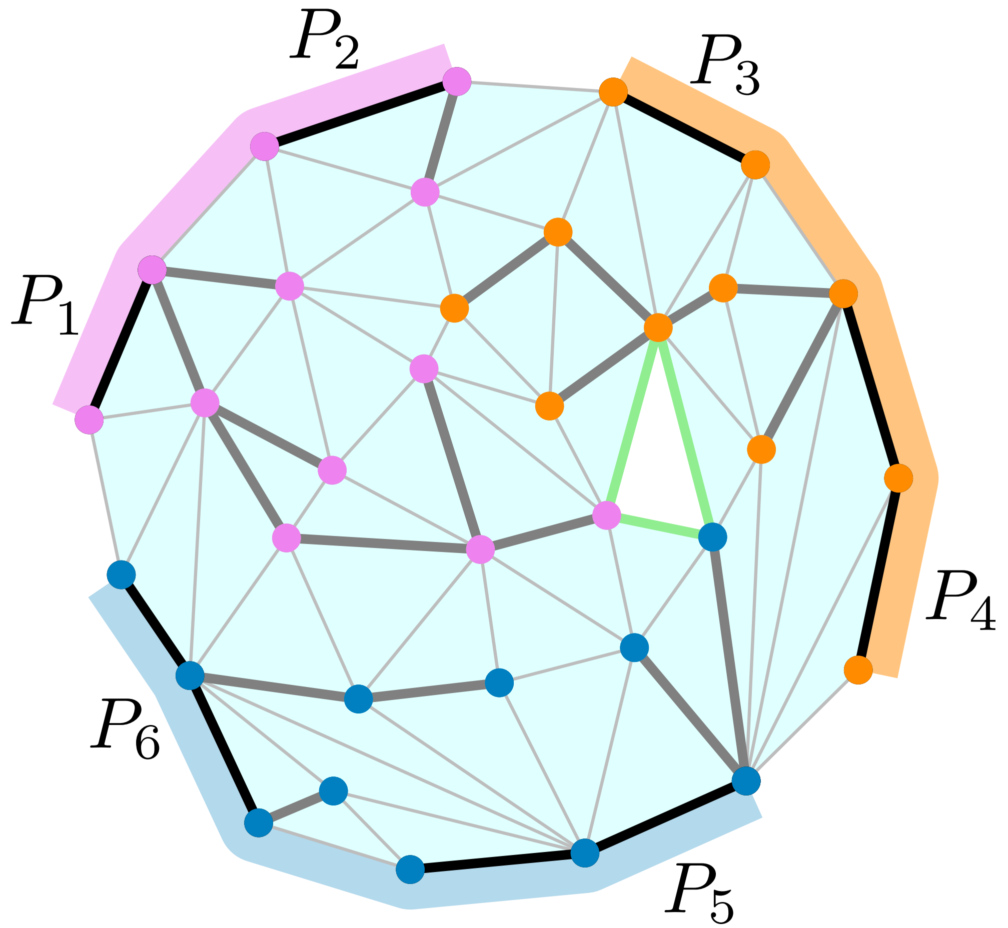

Proof of the **Main Lemma**.

▷ consider the parts of T in G

▷ exactly 3 groups of at most 2 paths each

▷ 3-coloring of $V(G)$ by going along T

▷ 3-colored inner face by Sperner's Lemma

The Main Lemma

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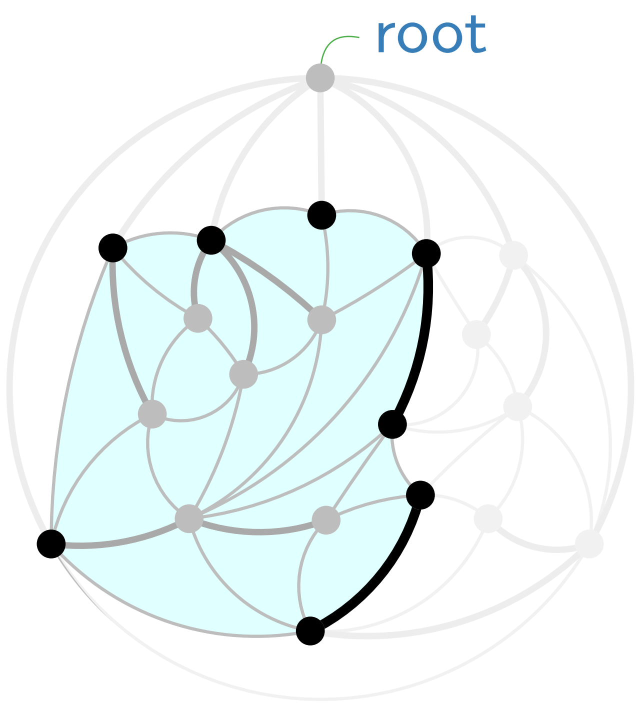

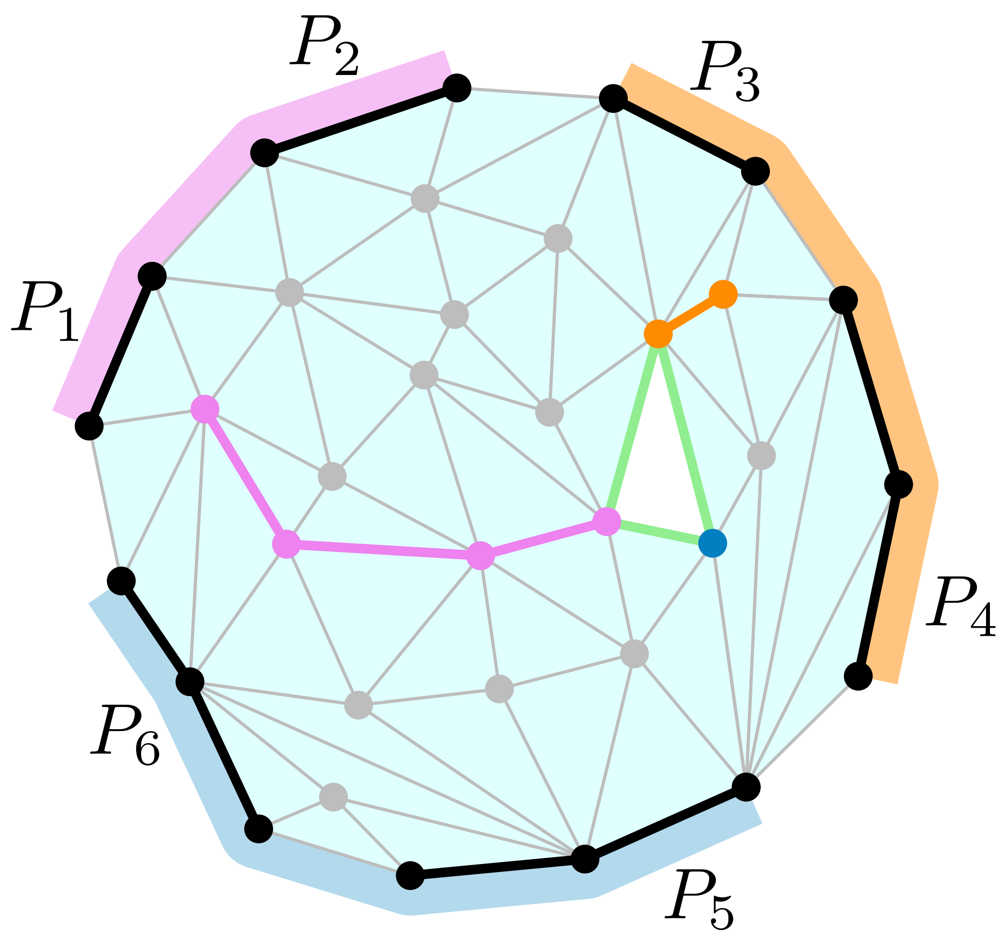

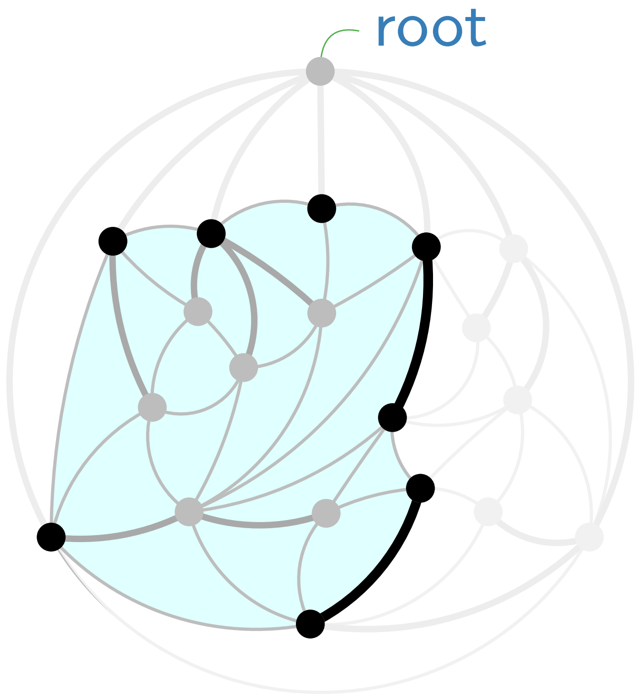

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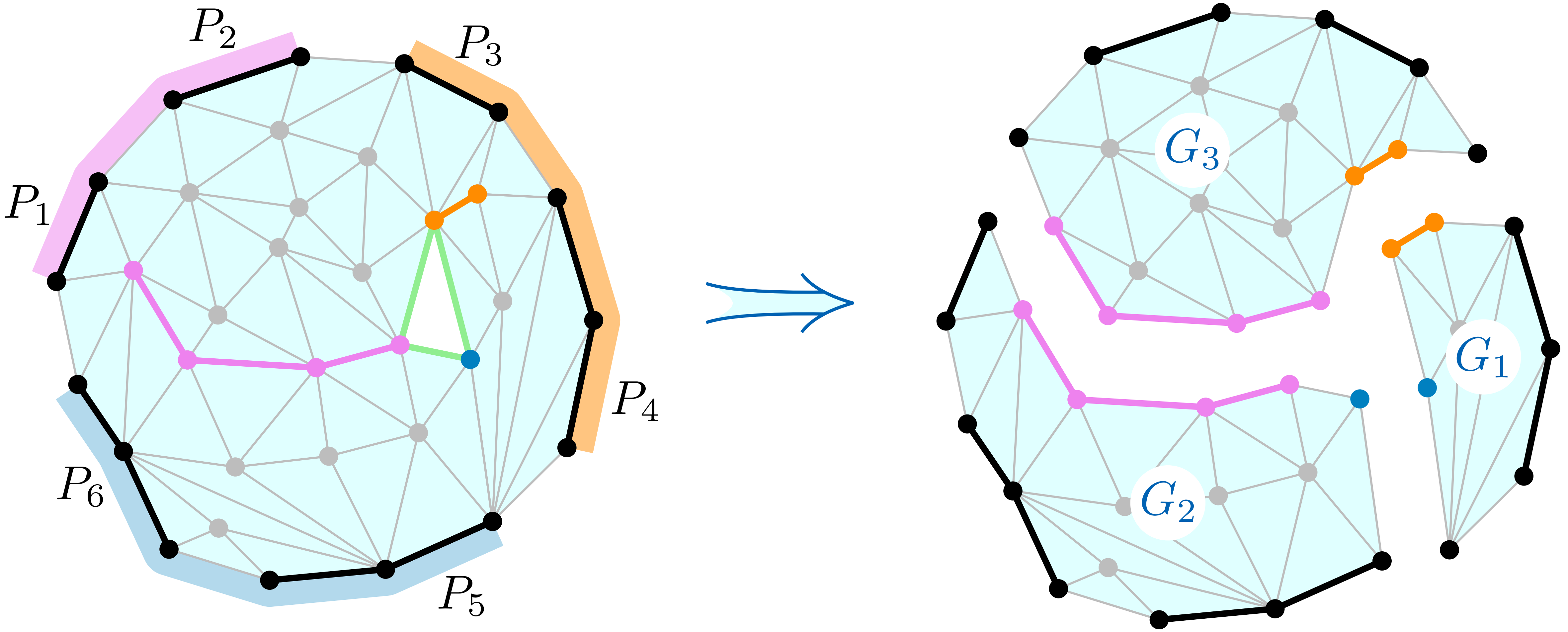
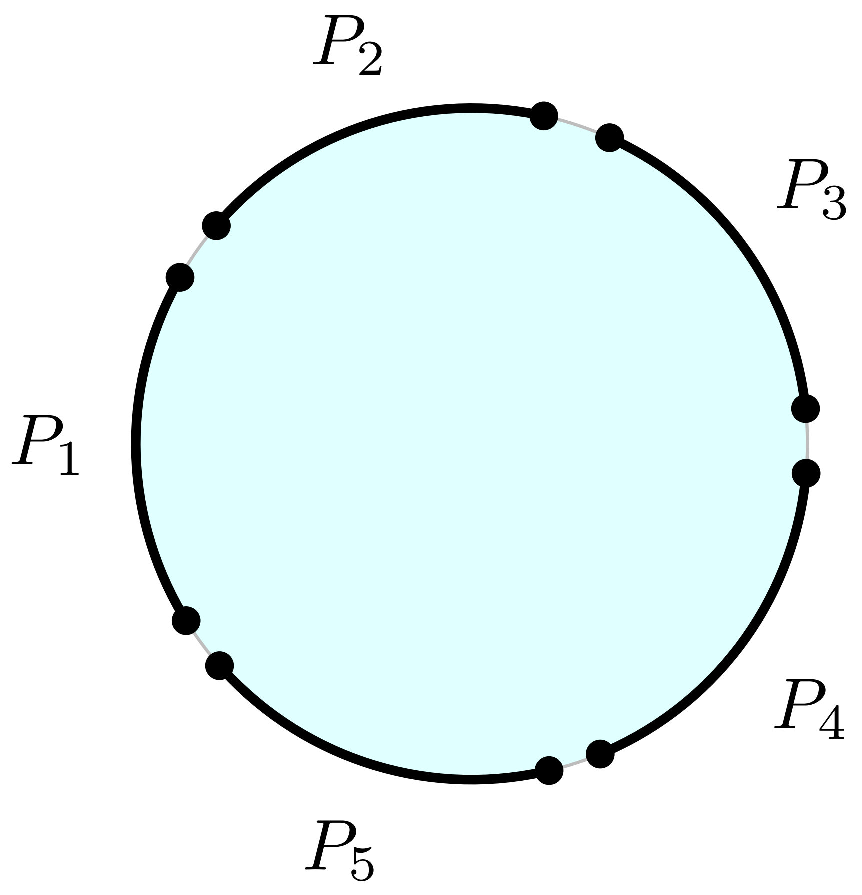


Improvement to $k \leq 5$ paths



- ▷ maintain $P_1, \ldots, P_k$ pairwise disjoint vertical paths, $k \leq 5$

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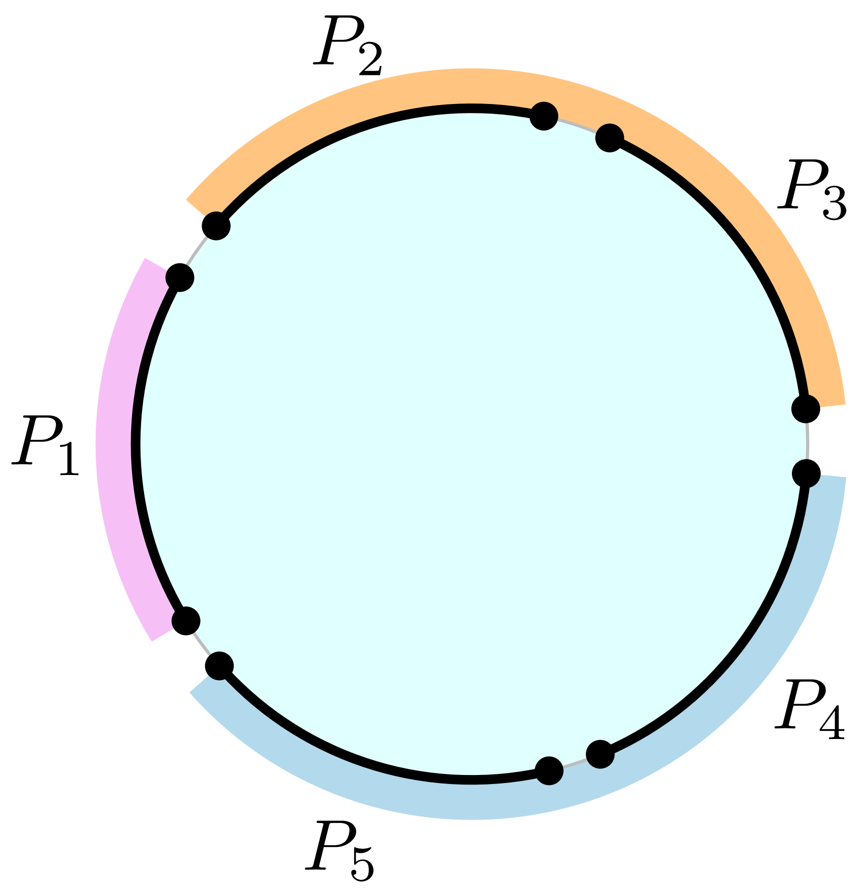

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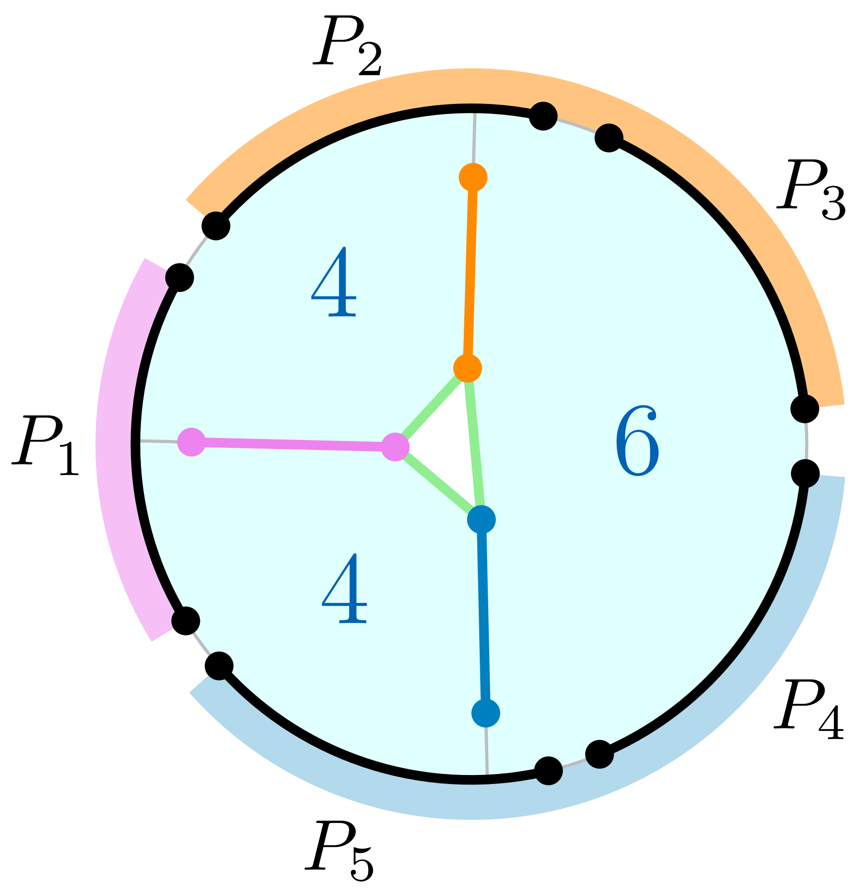

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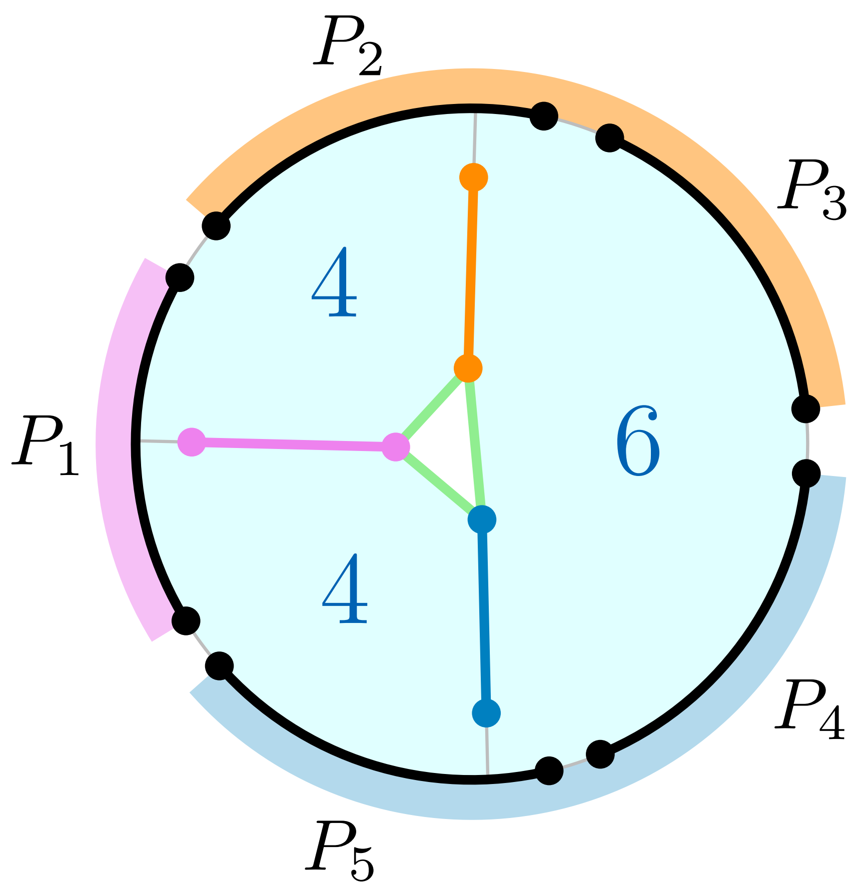

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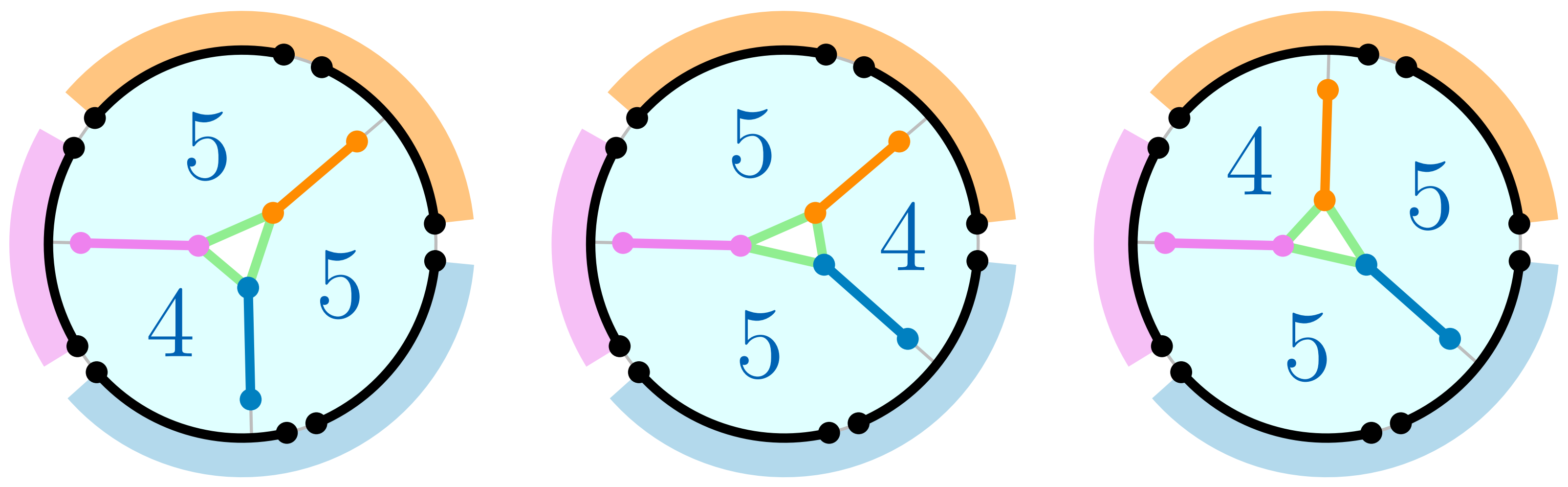

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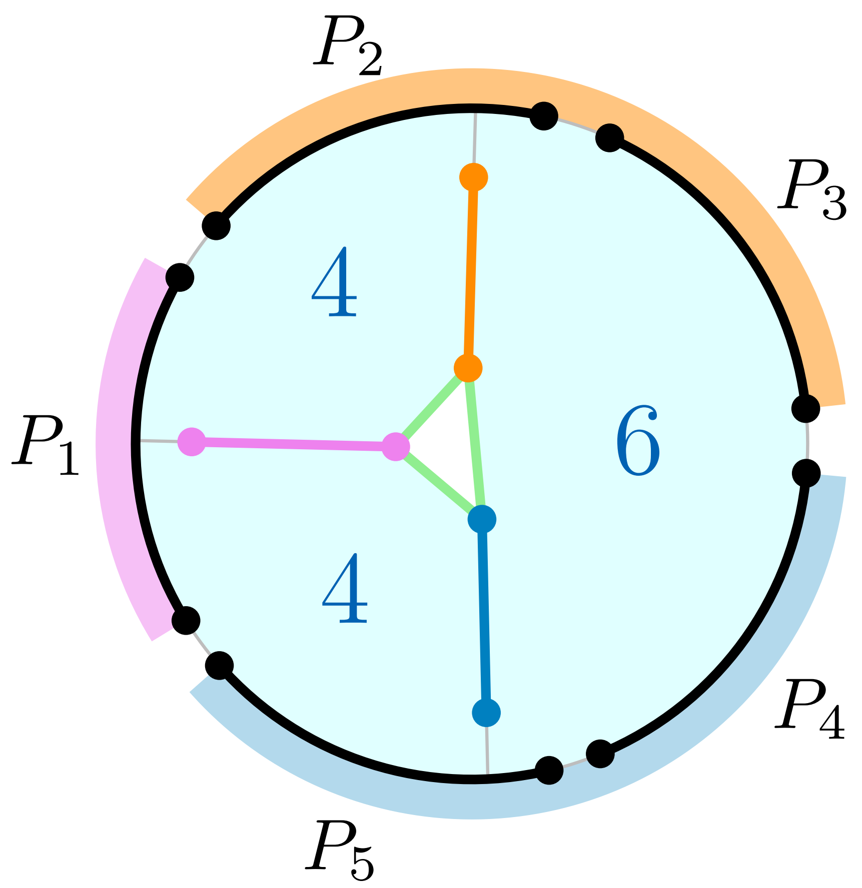

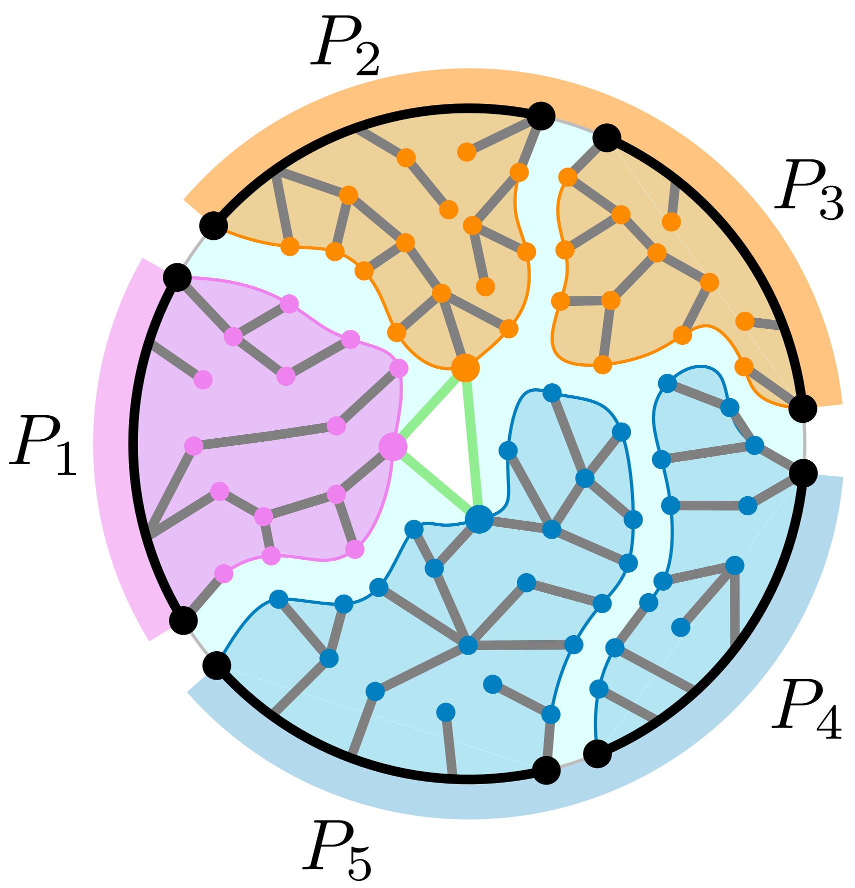

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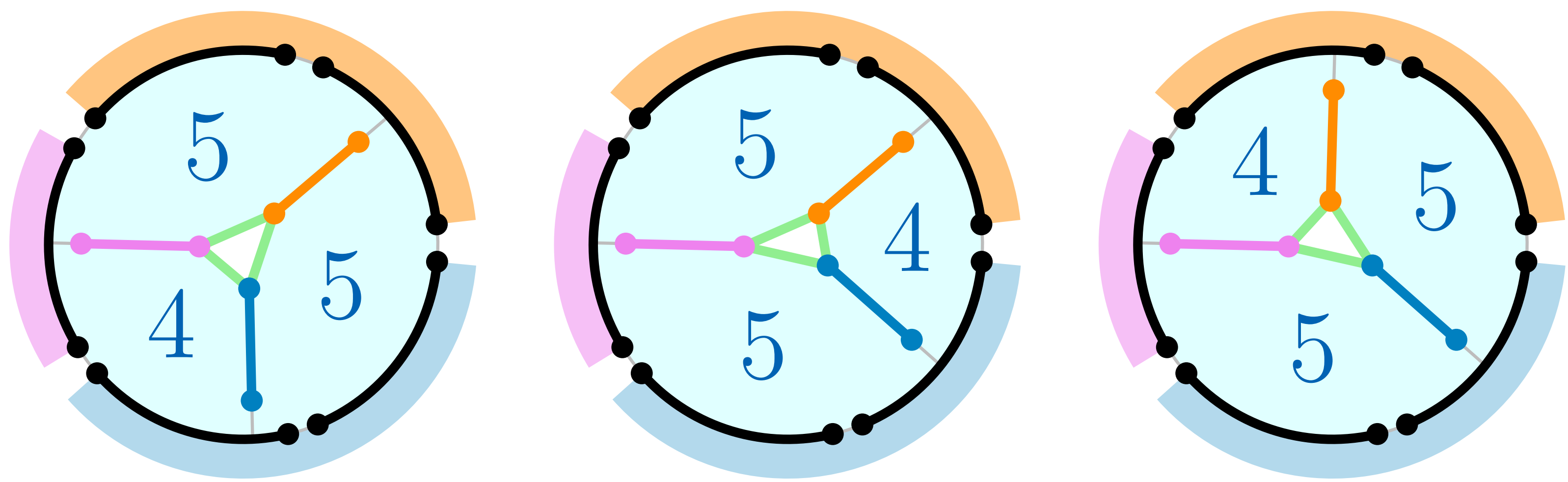
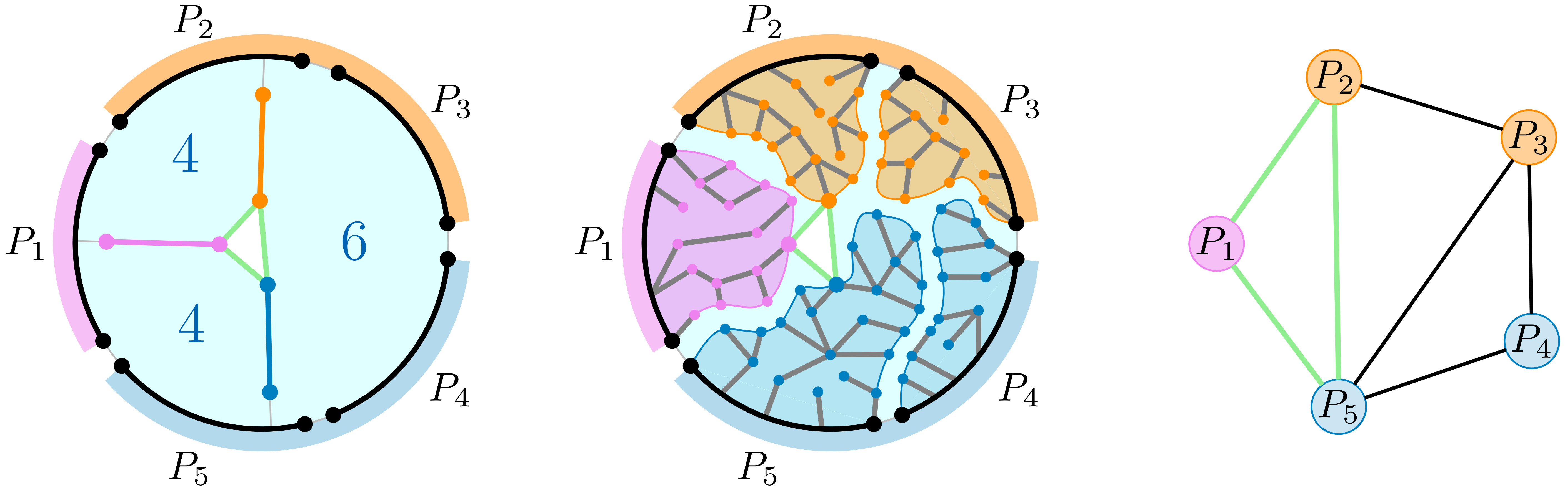
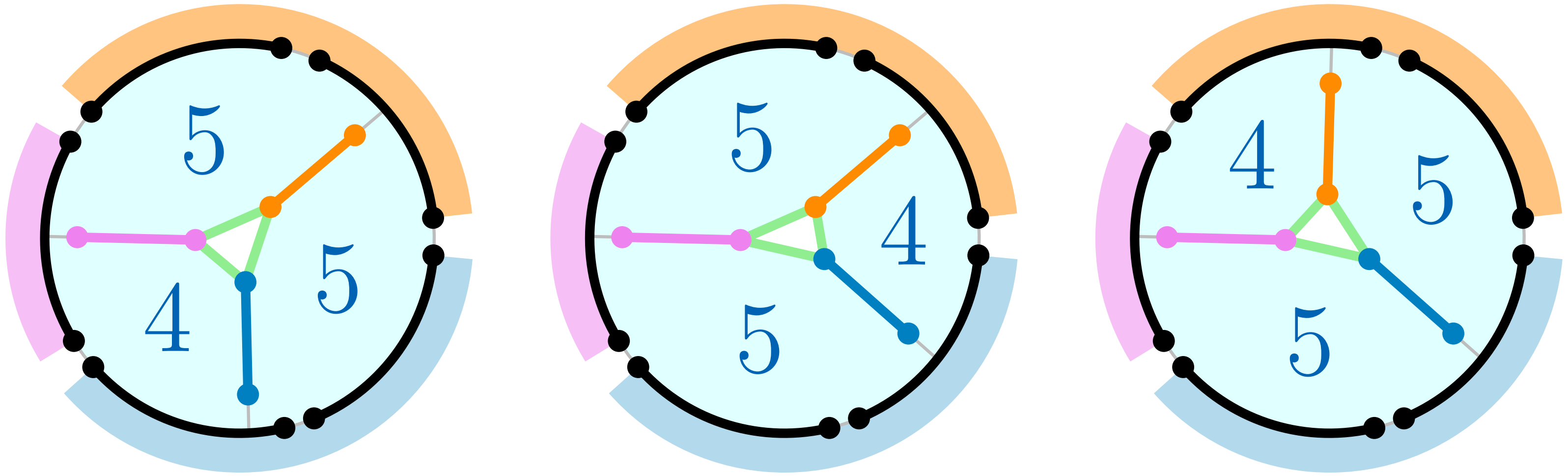


Improvement to $k \leq 5$ paths



- ▷ maintain $P_1, \ldots, P_k$ pairwise disjoint vertical paths, $k \leq 5$



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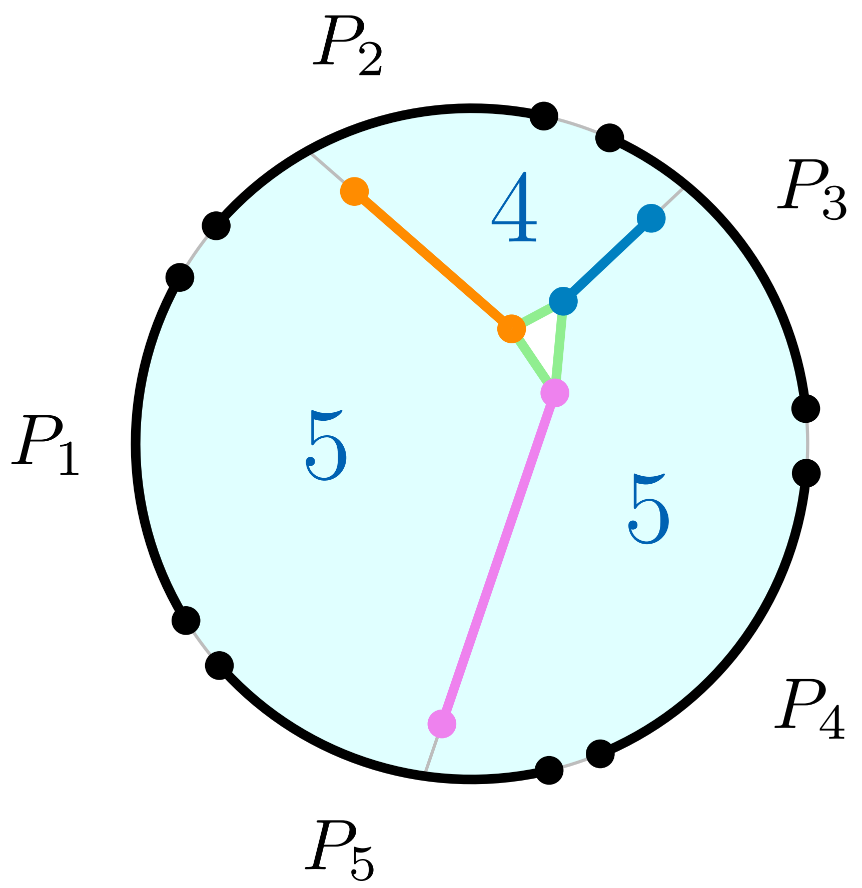
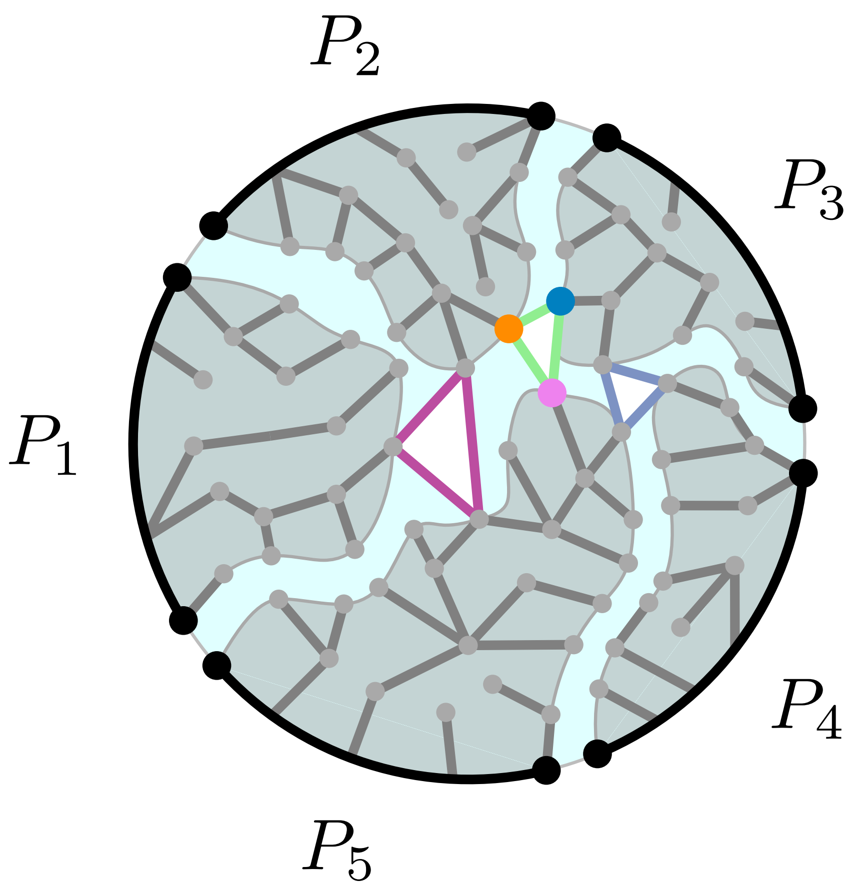
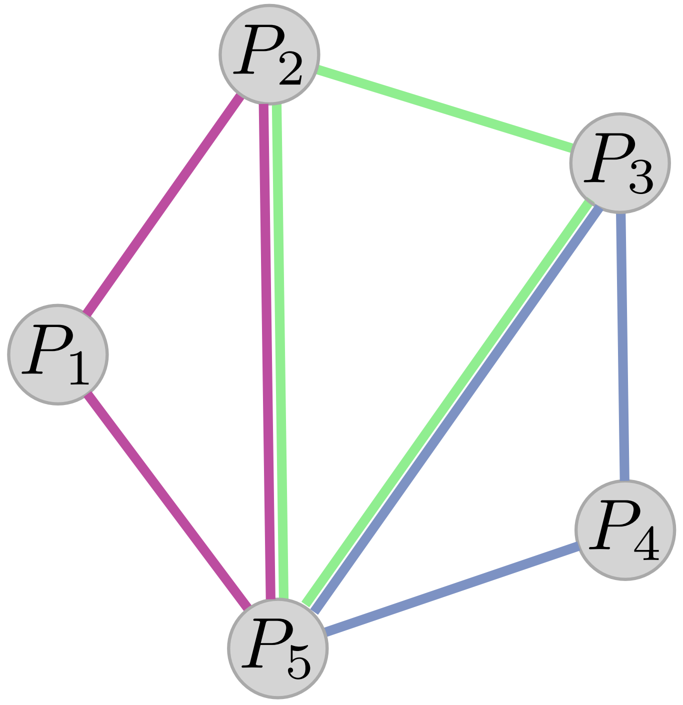






- maintain $P_1, \ldots, P_k$ pairwise disjoint vertical paths, $k \leq 5$
- take a Sperner triangle with at most 3 paths on each side

The Main Lemma

Let ▷ G^+ planar triangulation, T BFS tree rooted at an outer vertex

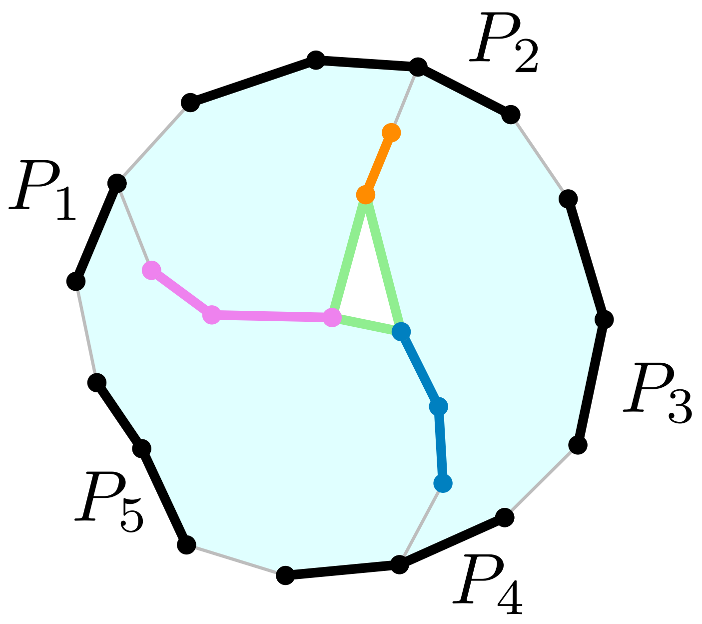
▷ $P_1, \ldots, P_k$ pairwise disjoint vertical paths, $k \leq 5$

▷ $F = [P_1, \ldots, P_k]$ cycle

▷ G near-triangulation on all vertices on and inside F

Then ▷ there exists a partition $\mathcal{P}$ of G into vertical paths with $P_1, \ldots, P_k \in \mathcal{P}$

▷ $H = G/\mathcal{P}$ has tree-decomposition of width $(k+3)-1$ with a bag containing $P_1, \ldots, P_k$



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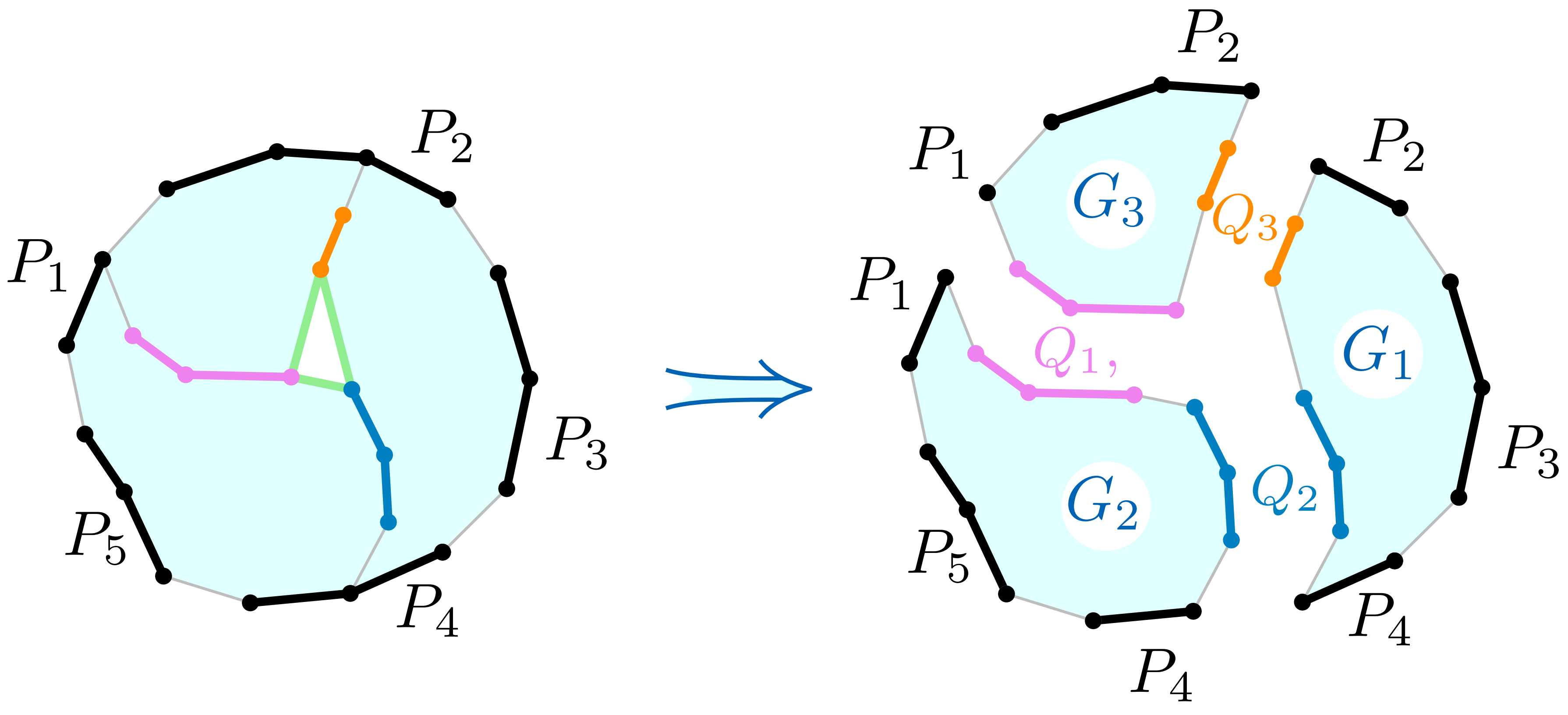

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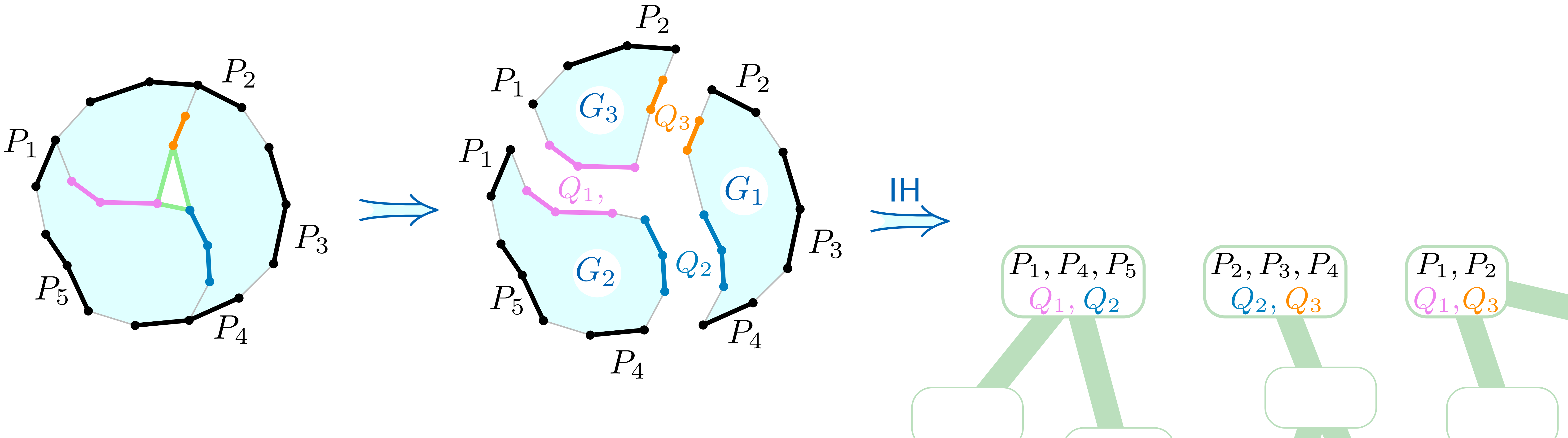

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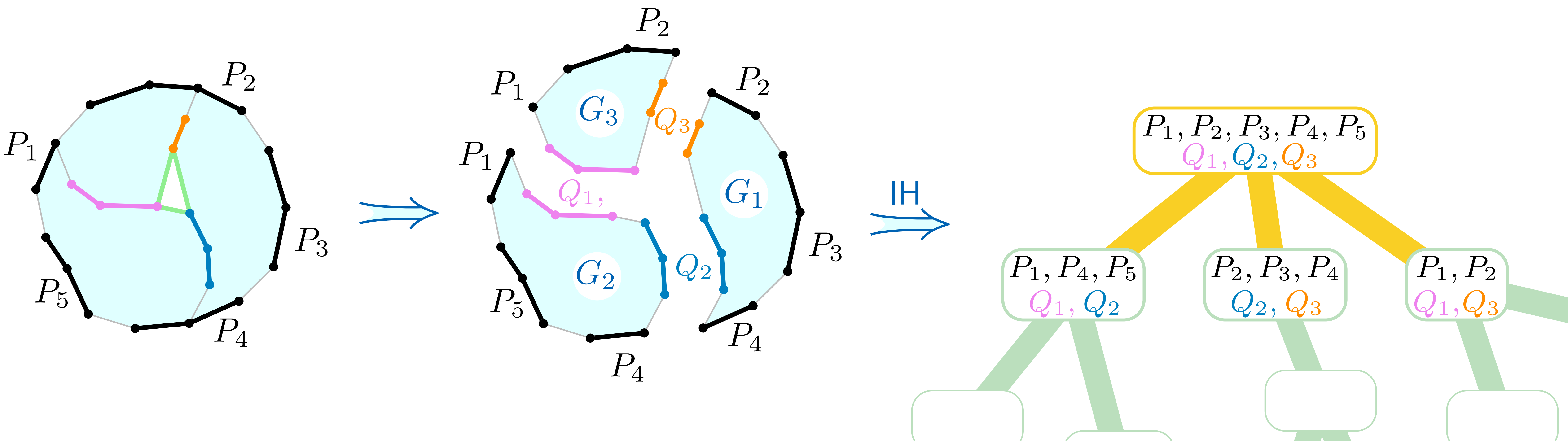

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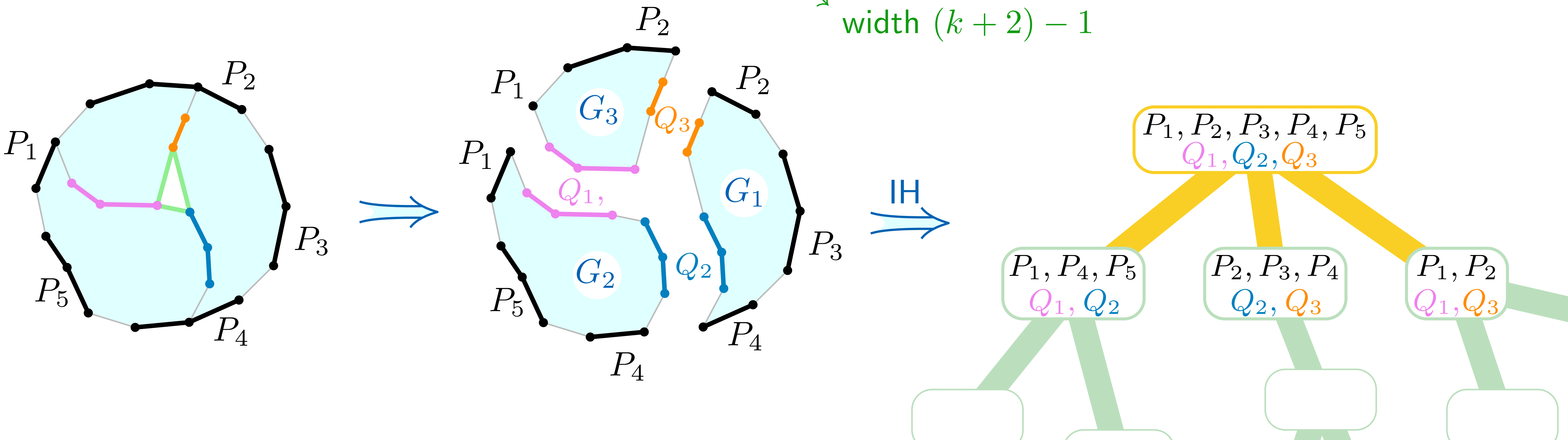
↘ width $(k+2)-1$



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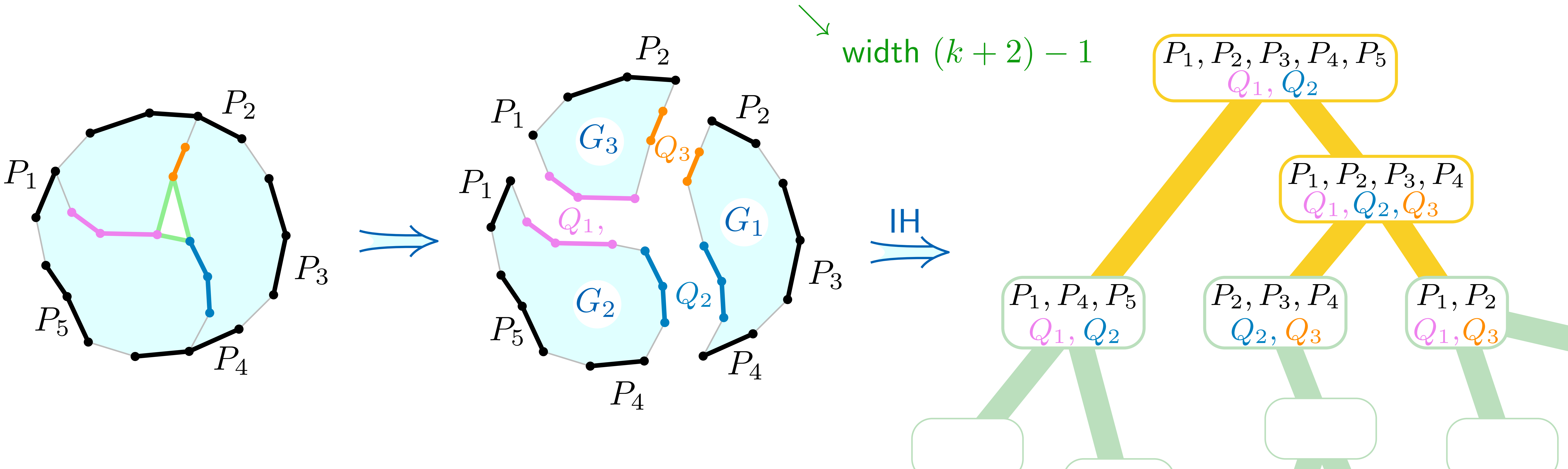
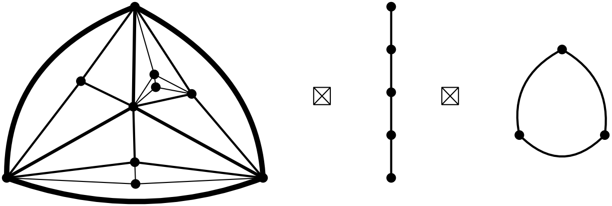


Second Version

Theorem (Dujmović-Joret-Micek-M-Ueckerdt-Wood 2019):
For every planar graph G there exists a planar graph H of treewidth at most 3 such that $G \subseteq H \boxtimes P \boxtimes K_3$.



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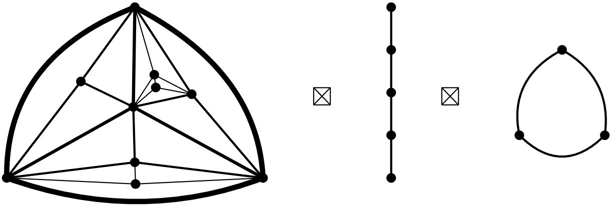
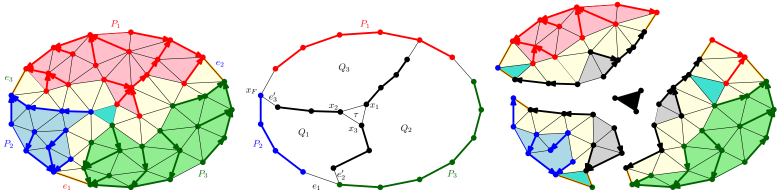


Useful when the (simple) treewidth of H is important

planar and treewidth-3 $\iff$ simple treewidth 3

Algorithmic Version

Theorem (M 2021): There exists an $O(n \log n)$ time algorithm that, given an n-vertex planar triangulation G finds H and P and the mapping $V(G) \rightarrow V(H \boxtimes P)$.

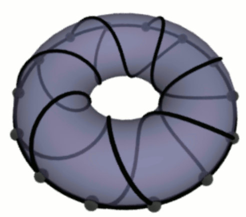


https://github.com/patmorin/lhp

Generalizations

Similar* product structure theorems for

- graphs of bounded genus and apex-minor free graphs (Dujmović-Joret-Micek-M-Ueckerdt-Wood 2019);



*$G \subseteq H \boxtimes P$, only the treewidth of H changes

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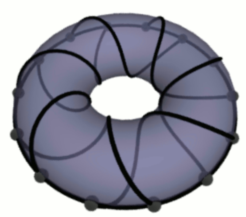

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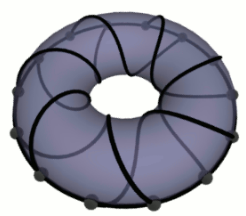

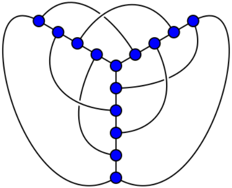

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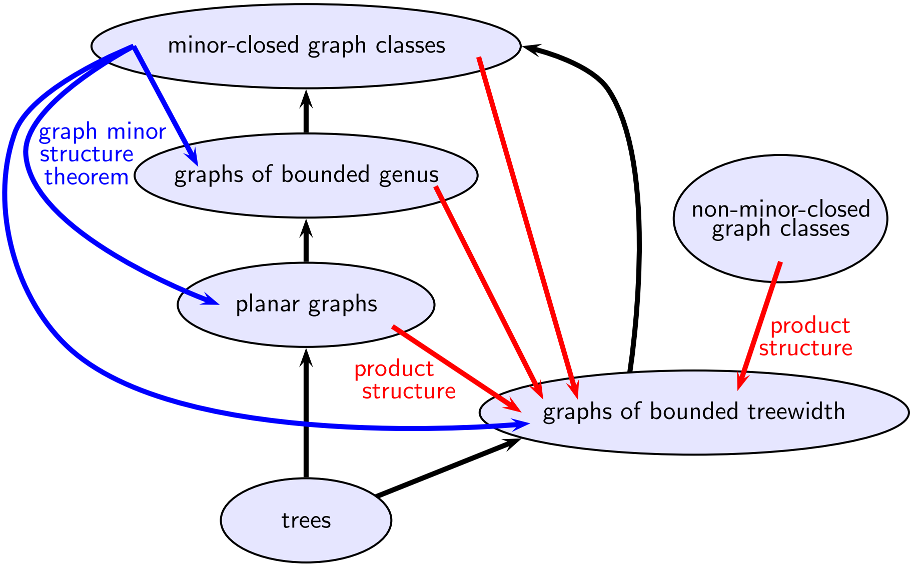
product structure



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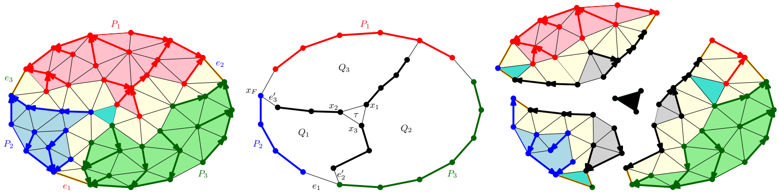

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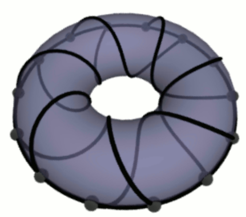

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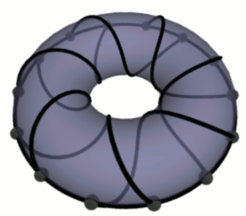

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Endings

- $P \boxtimes H$ planar graphs $3 \leq tw(H) \leq 6$
- What other classes of graphs have product structure?
- other applications