Graph Product Structure Theory

$G \subseteq H \boxtimes P$

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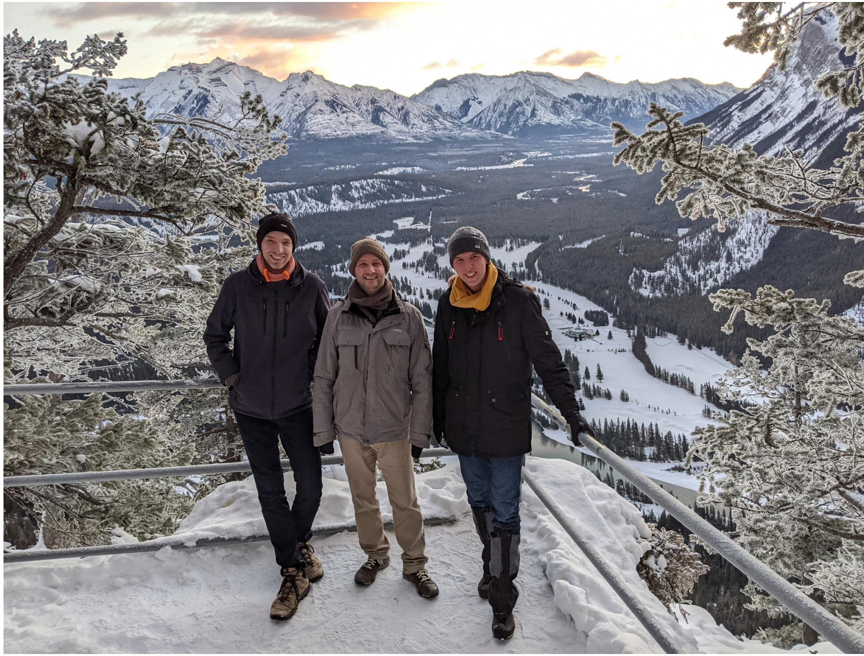


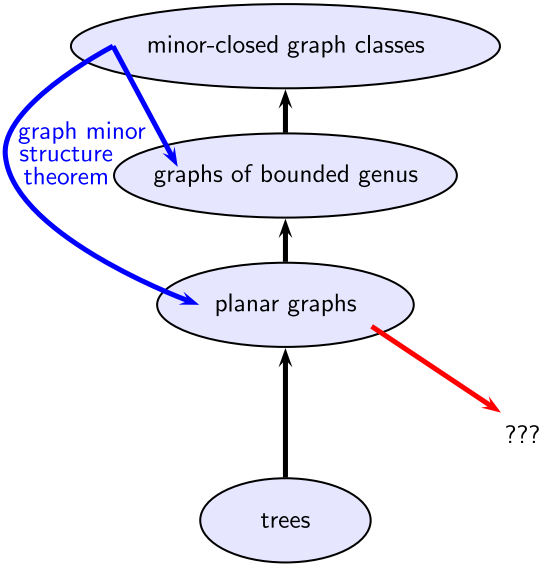
Ranking Graph Classes by Complexity

Simple

- paths (forests of paths)
- trees (forests)
- k-Trees (graphs of treewidth at most k)
- $\vdots$
- planar graphs
- $\vdots$
- proper-minor closed families
- $\vdots$
- bounded expansion
- $\vdots$
- all graphs

Complicated

structure of planar graphs



The Product Structure Theorem for Planar Graphs

Theorem (Dujmović-Joret-Micek-M-Ueckerdt-Wood 2019): For every planar graph G, there exists a planar graph H of treewidth at most ~~8~~ and a path P such that G is a subgraph of $H \boxtimes P$.

orin

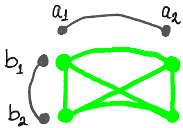
simple treewidth 6

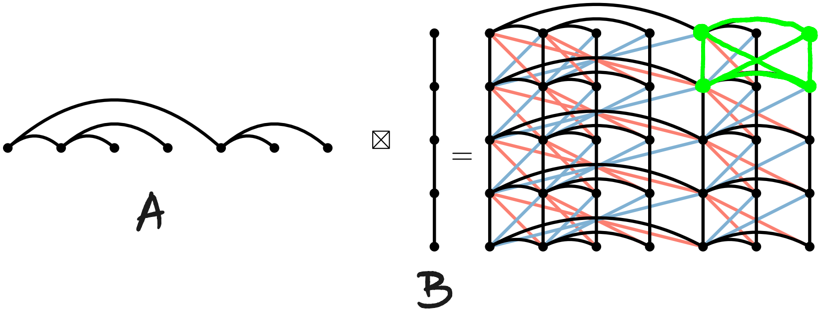
[Ueckerdt, Wood, Yi, 2021]

The Strong Graph Product $\boxtimes$

For two graphs A and B, the *strong product* $A \boxtimes B$ is a graph:

- $V(A \boxtimes B) := V(A) \times V(B)$
- (a_1, b_1) and (a_2, b_2) are adjacent if and only if:
  - $a_1 = a_2$ and $b_1 b_2 \in E(B)$;
  - $a_1 a_2 \in E(A)$ and $b_1 = b_2$; or
  - $a_1 a_2 \in E(A)$ and $b_1 b_2 \in E(B)$.

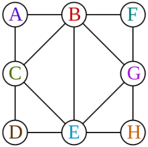


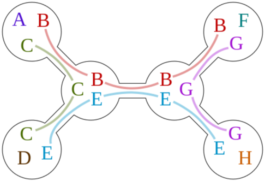


Treewidth

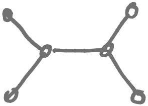
A *tree-decomposition* of a graph G represents each vertex as a subtree of a tree T so that the subtrees of adjacent vertices intersect in T

G





tree-decomposition of G



T

[Images courtesy of Wikipedia]

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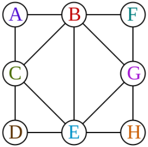
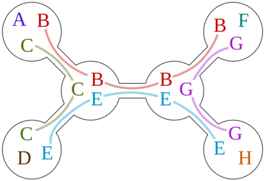


width := maximum bag size -1



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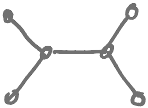

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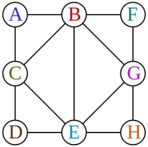

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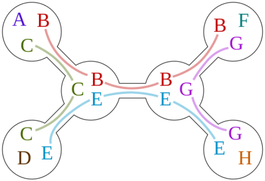
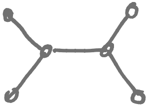


tree-decomposition of G



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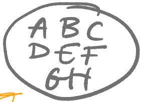

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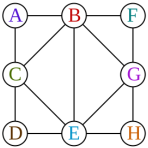
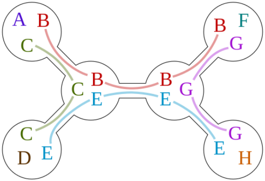


width := maximum bag size -1

treewidth := min width of tree-decomposition of G

→ measure of how "tree-like" a graph is



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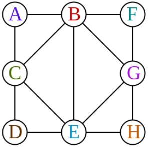

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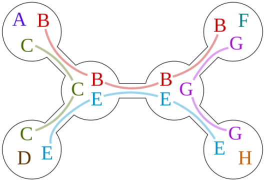
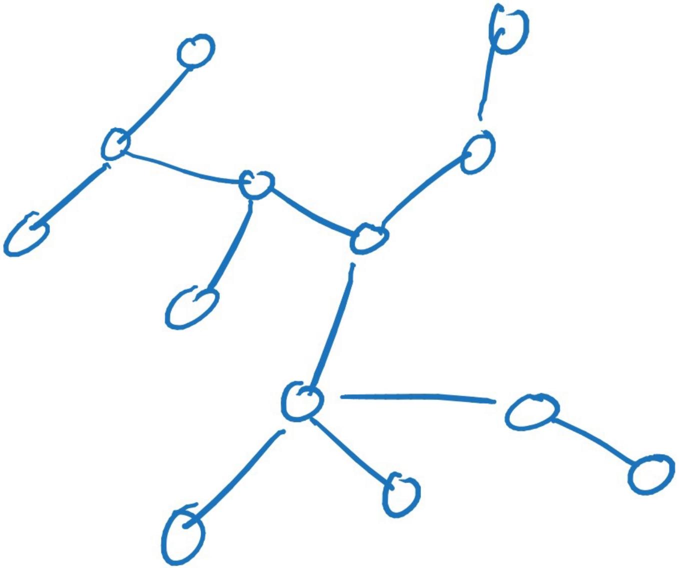


- trees?
- triangulations of polygons?
- grids?
- separators?

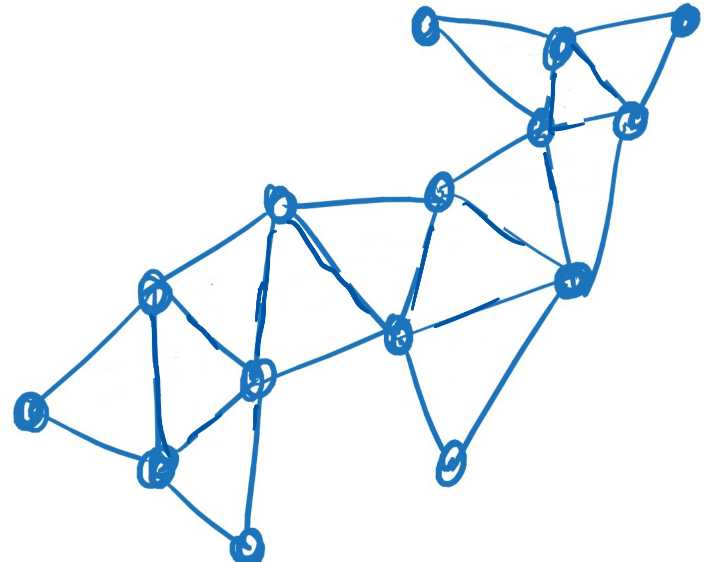
[Images courtesy of Wikipedia]

How do build tree-decomp

GOOD

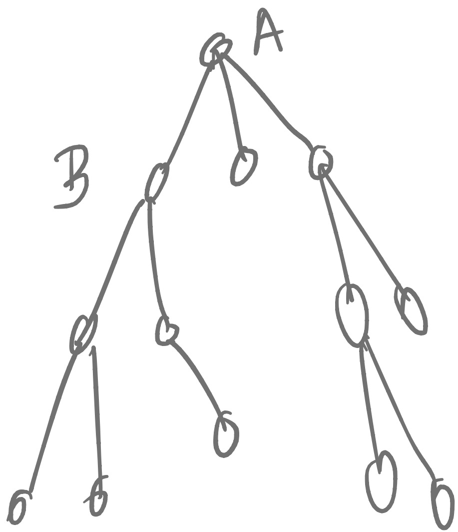


How would you build a tree-decomp. of this tree ?

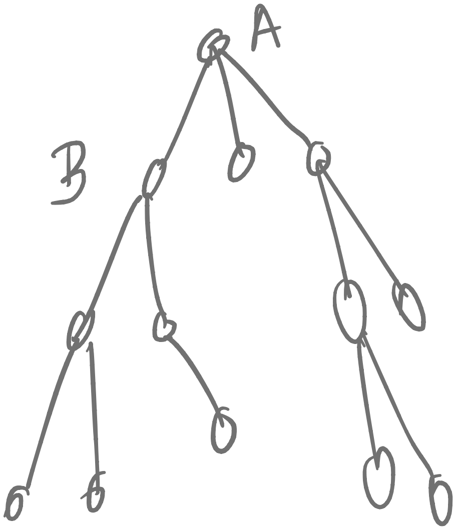


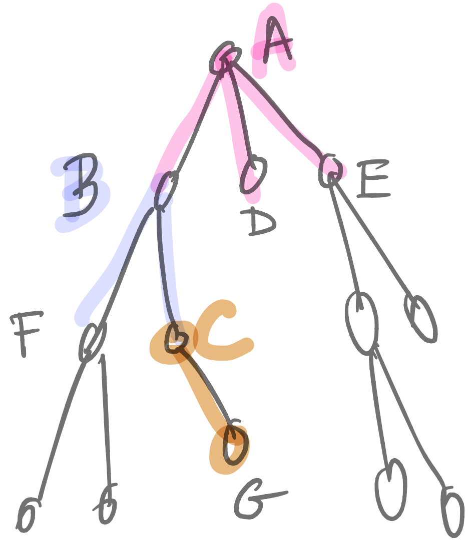
OR this outerplanar graph ?

trees

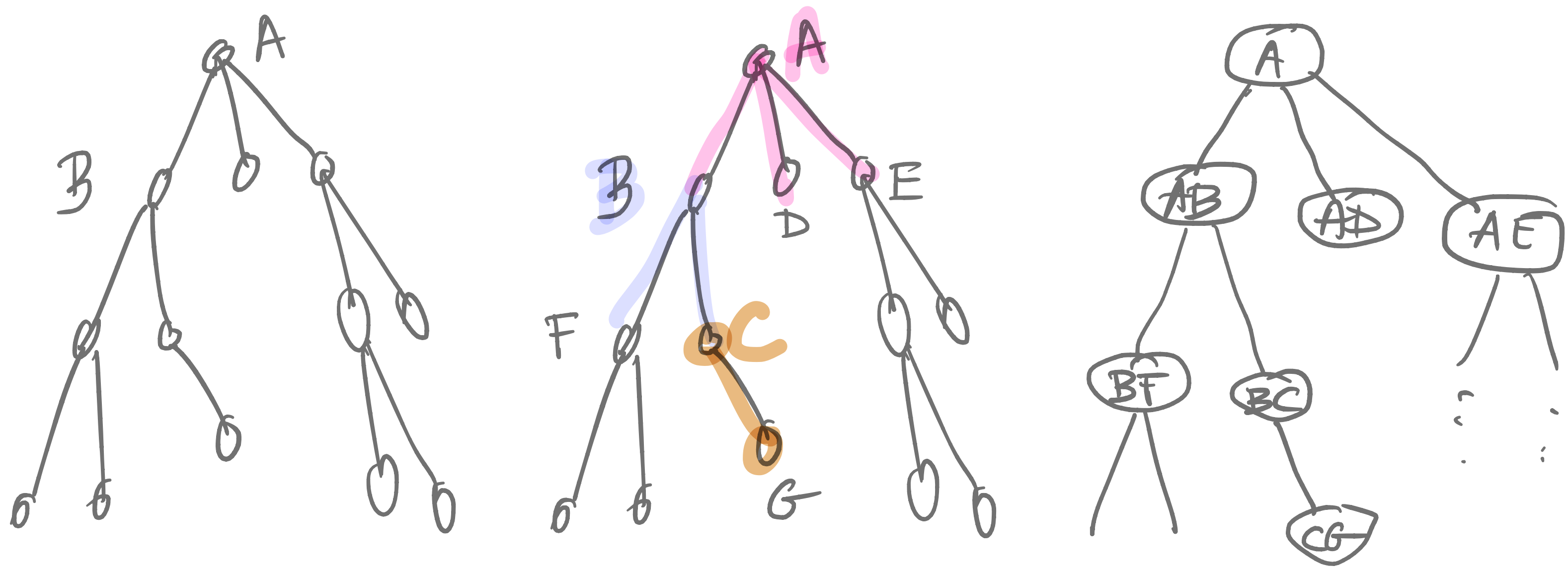


trees

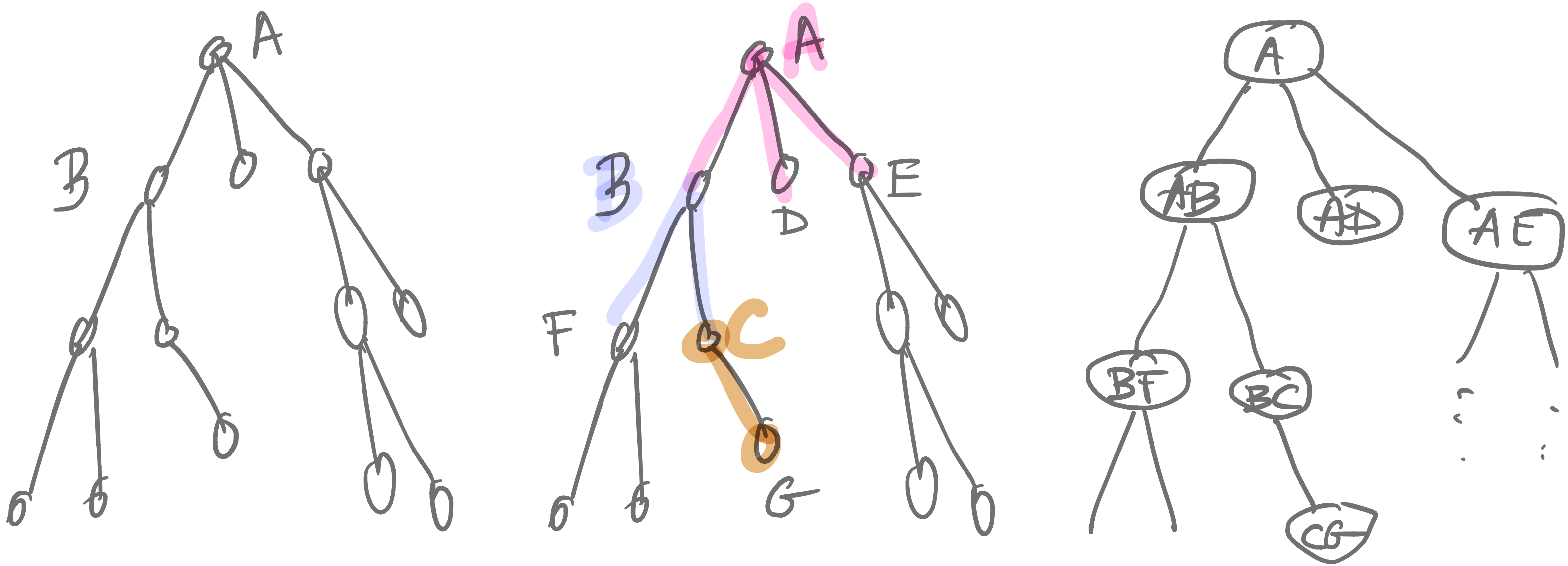




trees



trees

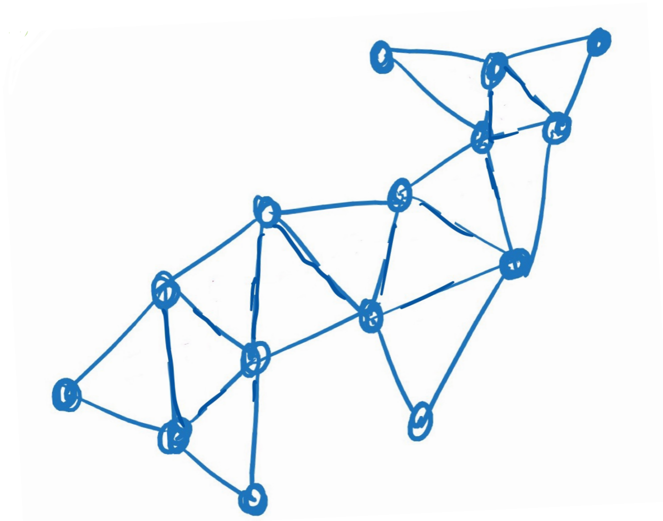


↓
width = 2

Trees have treewidth 1

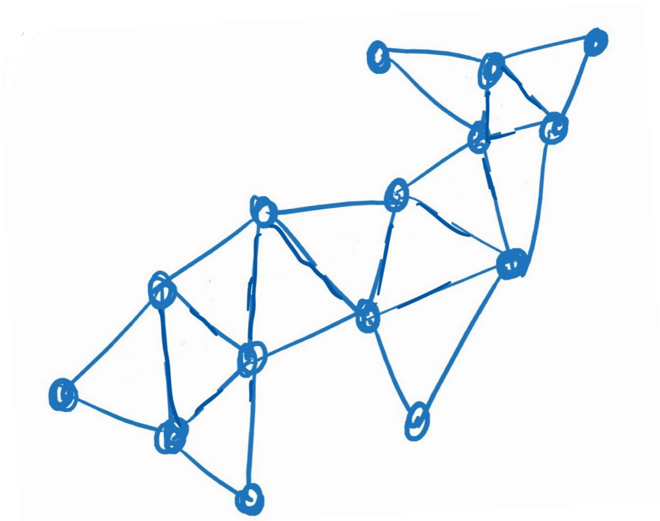
outerplanar graphs

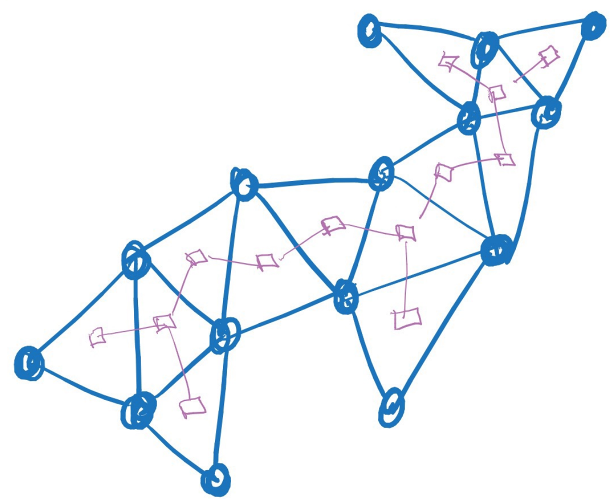
→ triangulations of polygons



outerplanar graphs

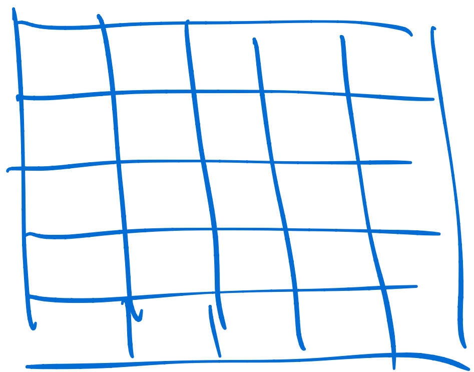
→ triangulations of polygons

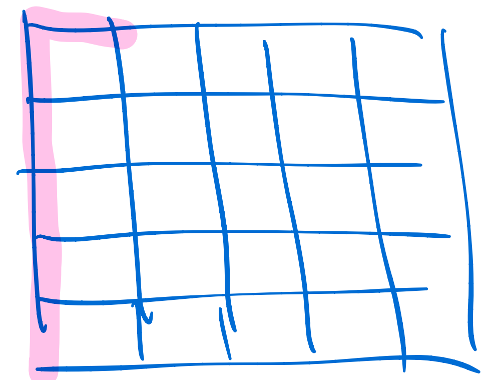


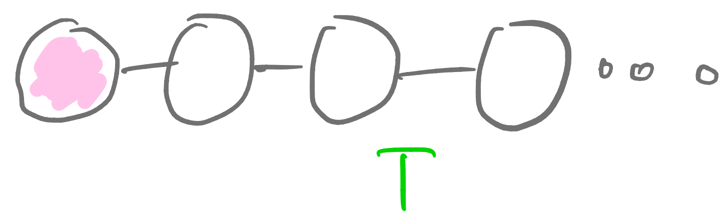


Th Outerplanar graphs have treewidth 2.

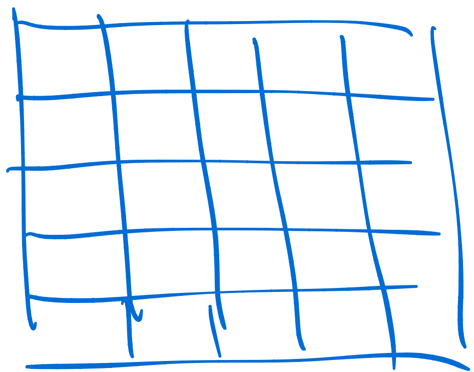
grids

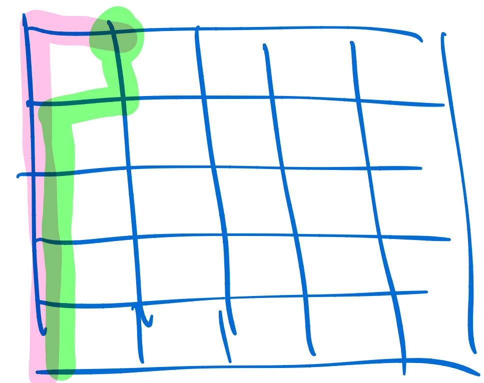


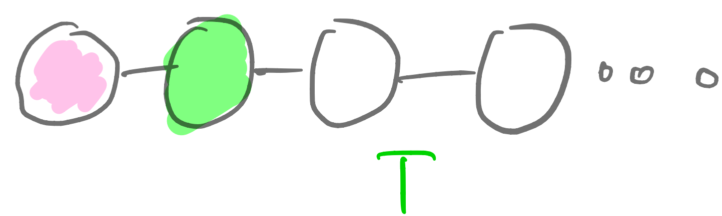




grids







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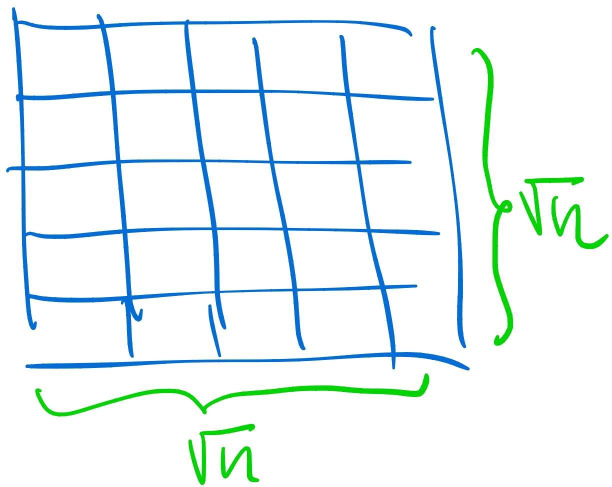
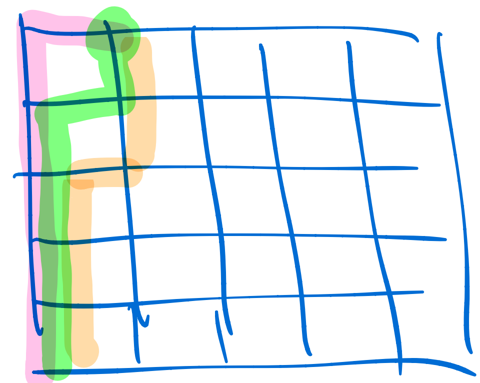
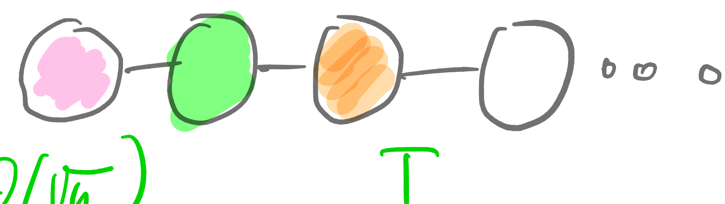




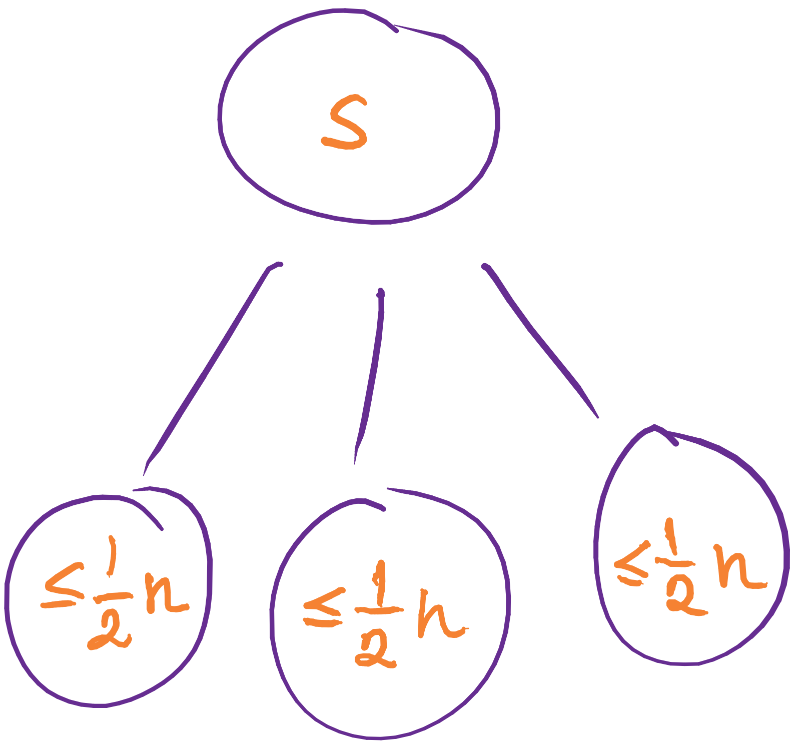
width $\Theta(\sqrt{n})$ $\longleftarrow$



Th// Treewidth of grids is $\Theta(\sqrt{n})$

CLASSICAL SEPARATORS

balanced $\alpha \in [\frac{1}{2}, 1)$



$S \subseteq V(G)$

s.t every connected component of $G - S$ has $\leq \frac{1}{2}|V(G)|$ vertices.

$|S| :=$ size of separator

MANY APPLICATIONS

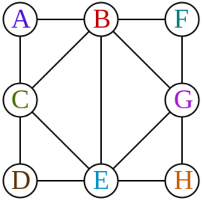
graphs with small separators

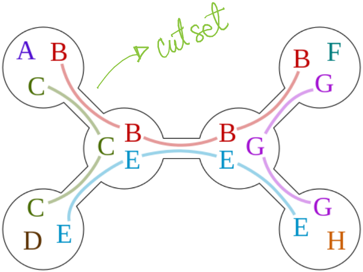
- parametrized (exact) algorithms
- approximation algorithms (divide and conquer)

graphs with no small separators

- K_n $\geq \frac{n}{2}$
- expanders (sparce graphs with high 'global' connectivity

Where are the separators?





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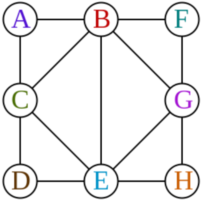
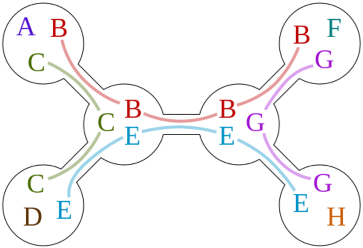




Robertson and Seymour:

balanced separators $\leq tw + 1$

for weighted graphs too

Dvořák and Norin:

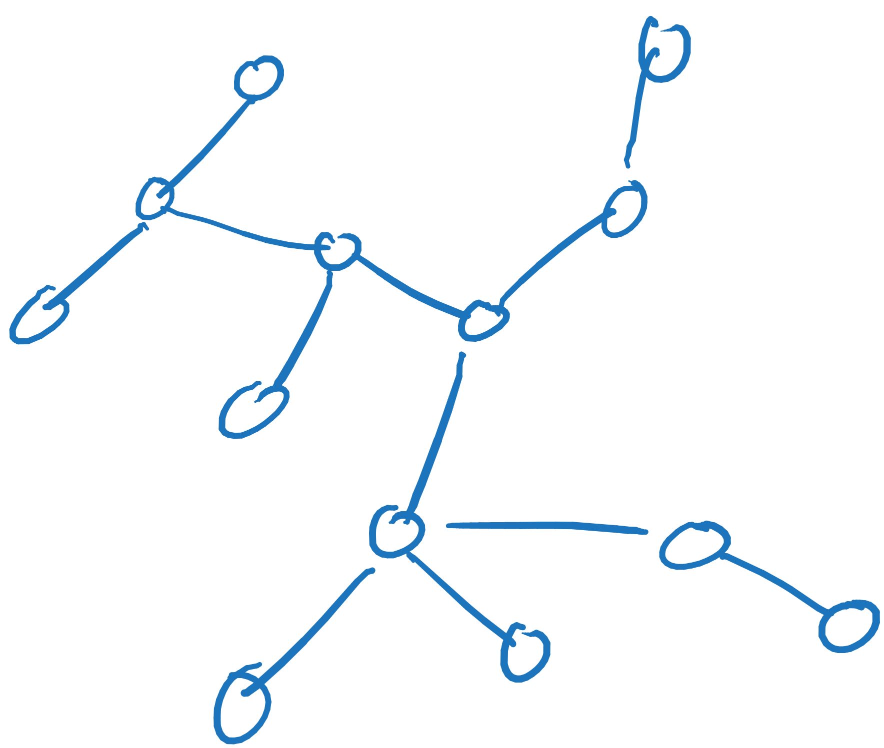
Reverse is true.

$$tw(G) \leq 105\, sep(G)$$

$\Rightarrow$ Treewidth and separ. tied

Example : TREES

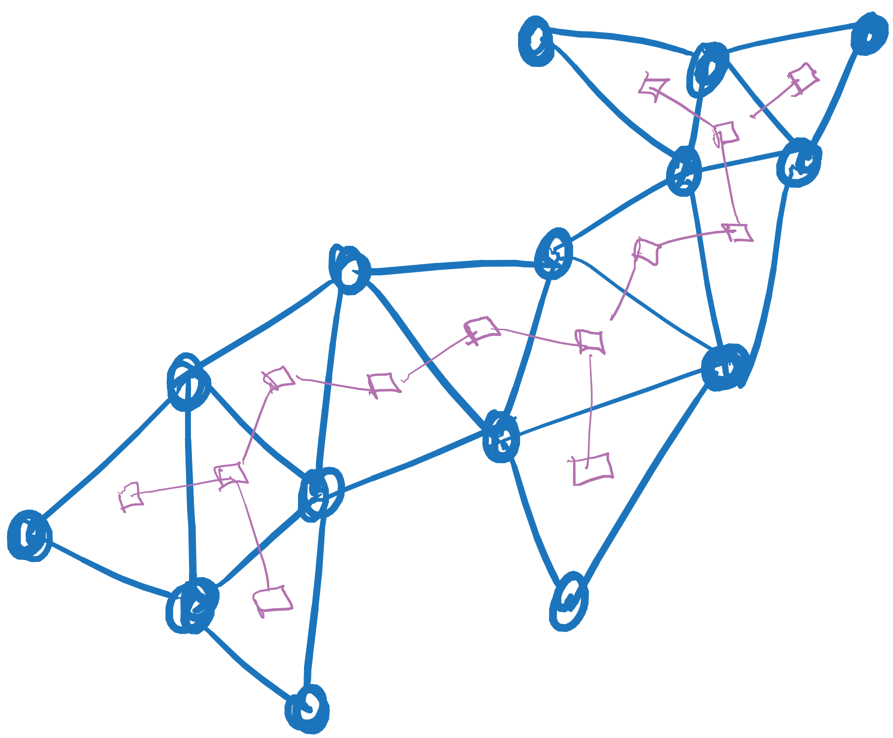
Th : Every tree has a separator of size 1.



Example : OUTERPLANAR

· outerplanar graph

Th: Every ~~tree~~ has a separator of size ~~1~~ 3.

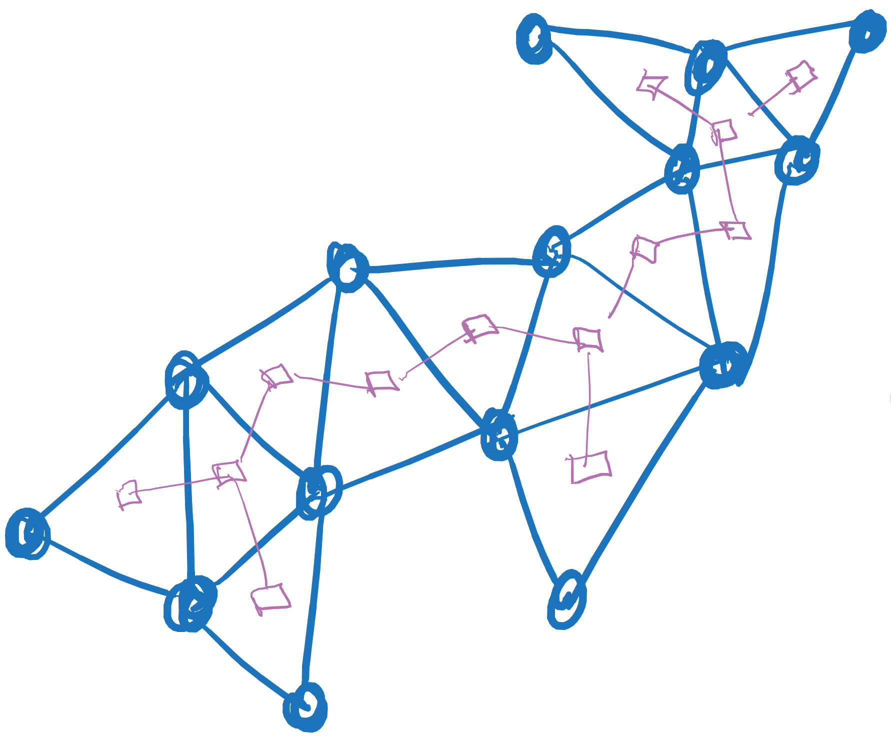


Example : OTHER

• outerplanar graph

Th: Every ~~tree~~ has a separator of size ~~1~~ 3.

→ • series-parallel too



Q: Do all planar graphs have constant size separators?

Example : OTHER

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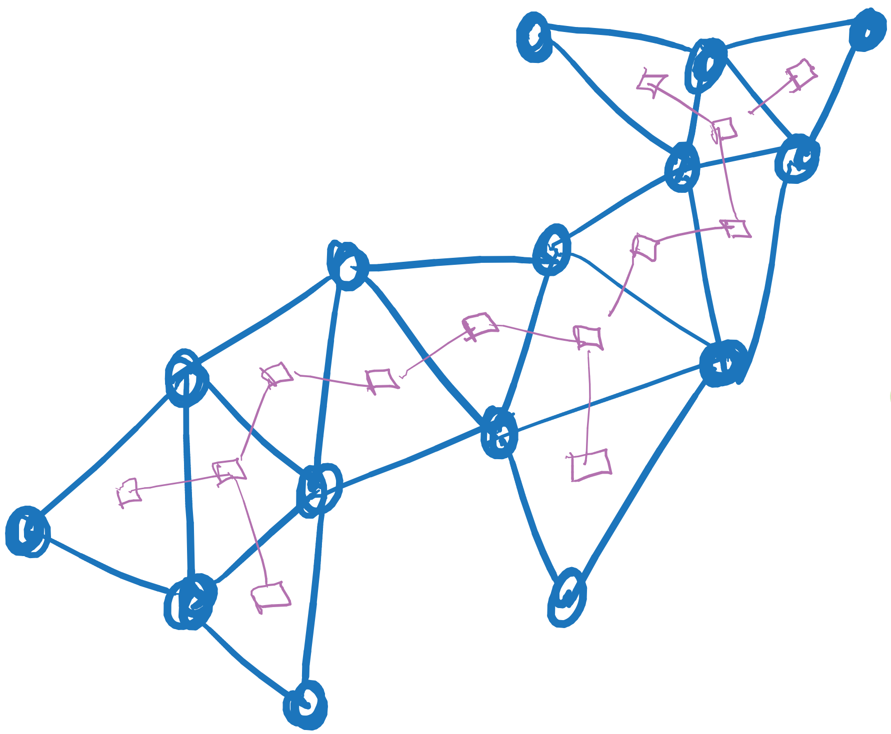
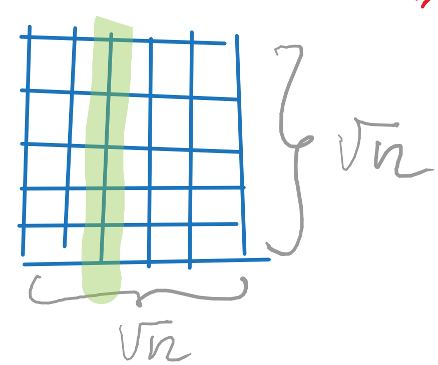


Q : Do all planar graphs have constant size separators?

NO



What about planar graphs?

Lipton & Tarjan '80s:

Planar graphs have $\Theta(\sqrt{n})$ separators.

→ bounded genus
→ proper minor closed

Used extensivelly:

- PTAS alg
- EPTAS
- bidimensionality

$(1+\varepsilon)$
$(1-\varepsilon)$

→ does not help us though
e.g nonrepetitive colourings,
queue layouts ...

The Product Structure Theorem for Planar Graphs

Theorem (Dujmović-Joret-Micek-M-Ueckerdt-Wood 2019):
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For every planar graph G, there exists a planar graph H of treewidth at most $\cancel{8}$ and a path P such that G is a subgraph of $H \boxtimes P$.

simple treewidth 6

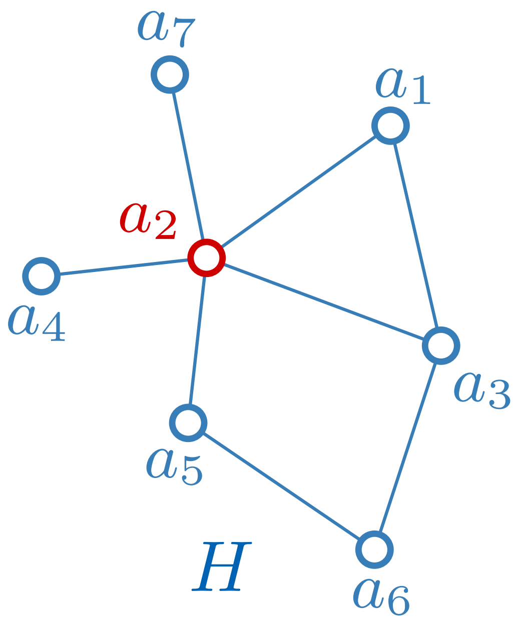
[Ueckerdt, Wood, Yi, 2021]

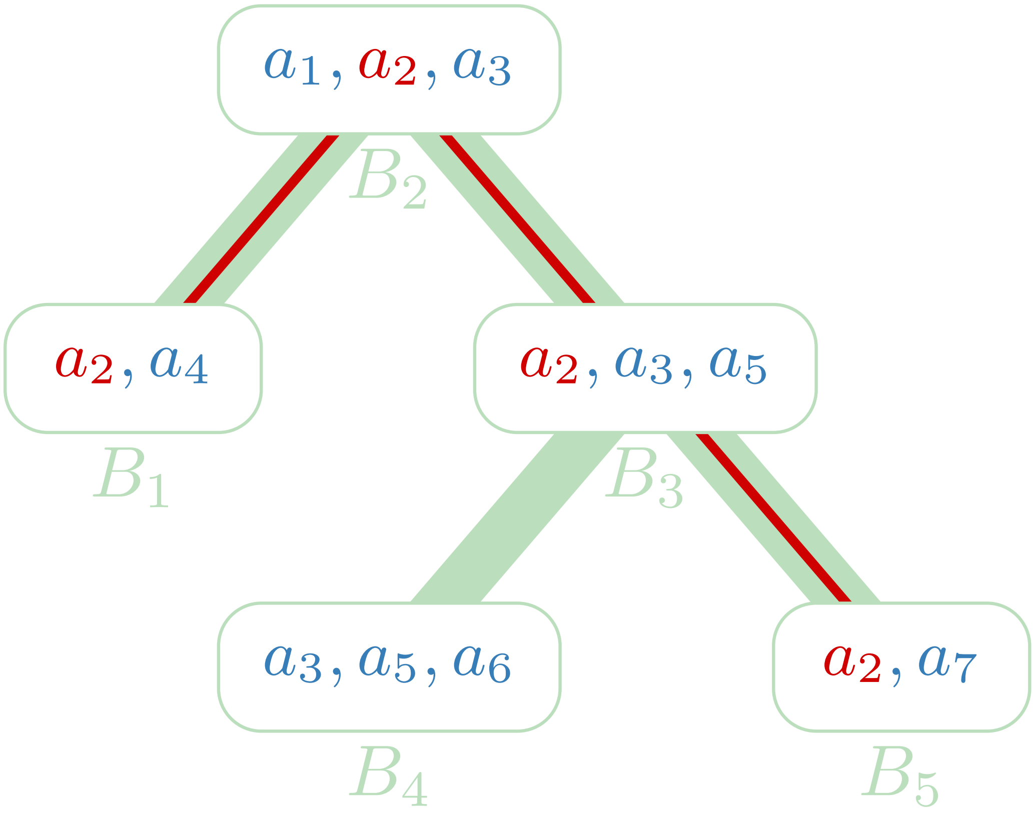
Tree Decompositions

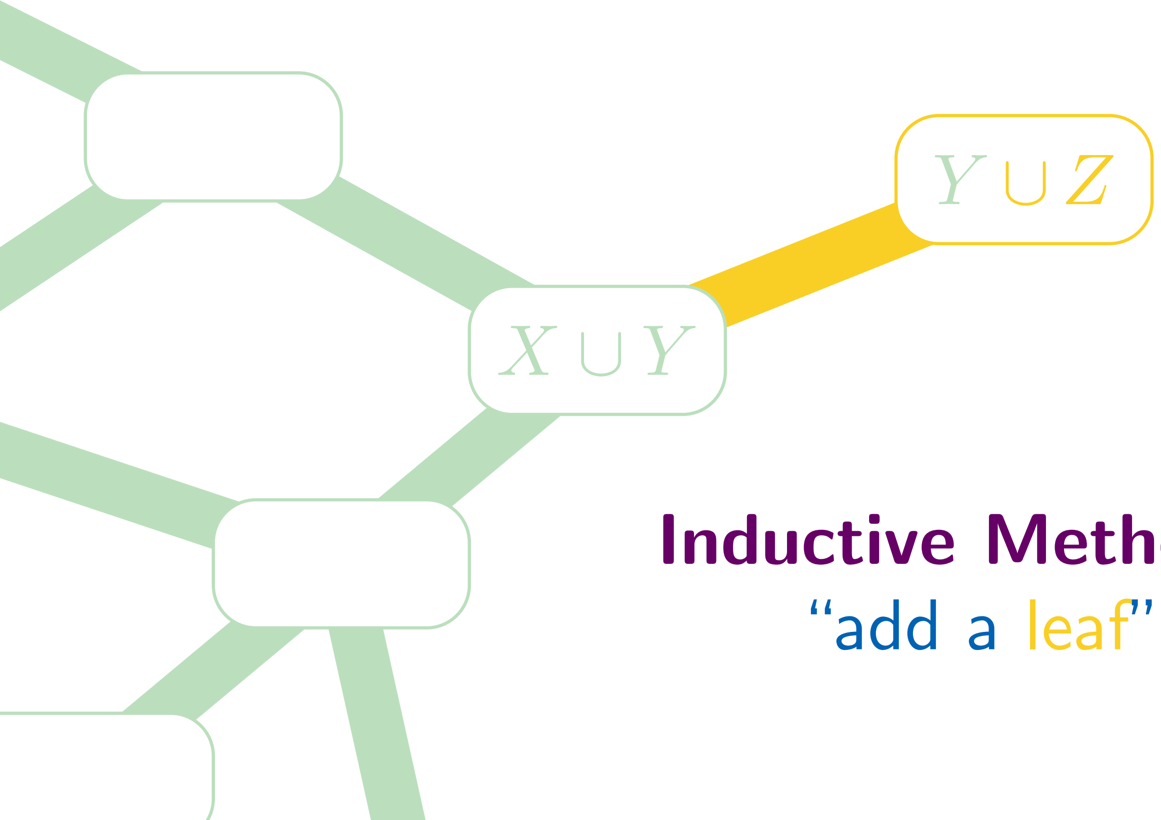
A tree decomposition of H are
vertex sets (bags) $B_1, B_2, \ldots$
such that

- ▷ $B_1, B_2, \ldots$ are the vertices of a tree
- ▷ $v \in V(H) \Rightarrow \{B_i \mid v \in B_i\}$ subtree
- ▷ $uv \in E(H) \Rightarrow \exists i : u, v \in B_i$

The width is the maximum size of a bag -1.

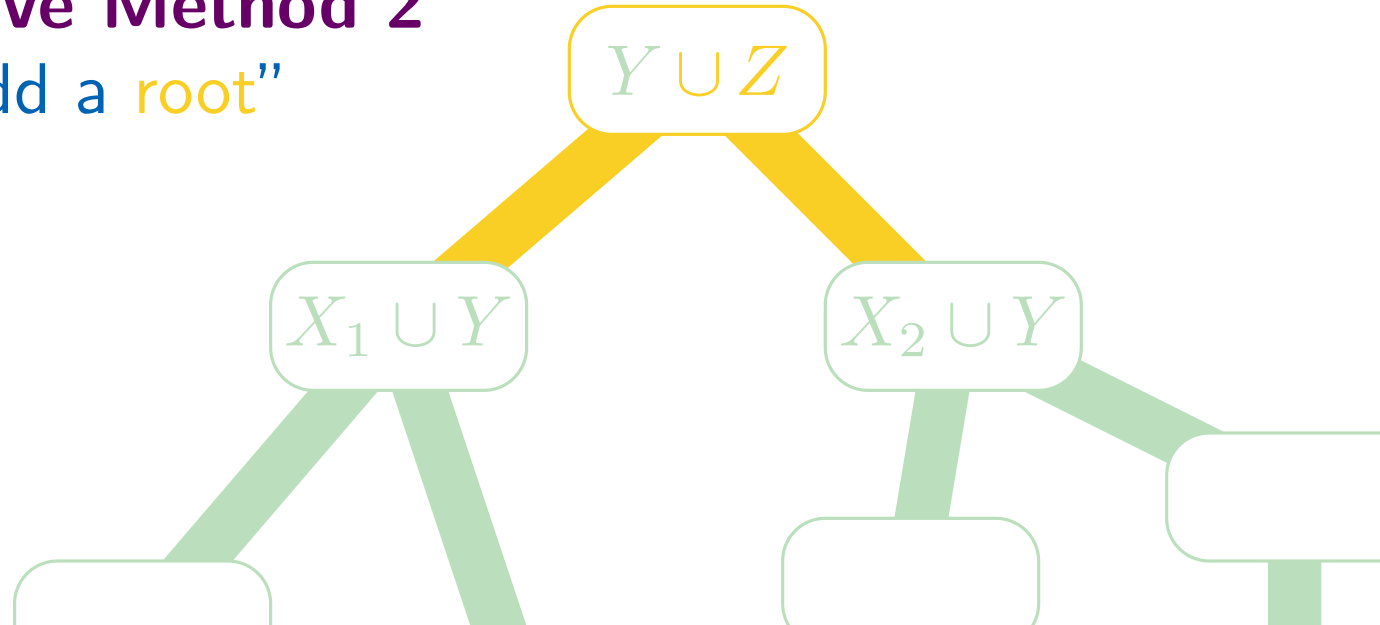






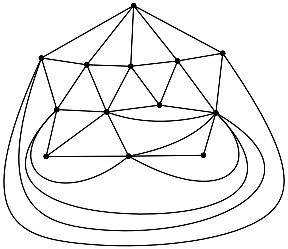
Inductive Method 1
"add a leaf"

Inductive Method 2
"add a root"



The Precursor

Theorem (Siebertz-Pilipczuk 2018): For any planar triangulation G, there exists a partition $\mathcal{P}$ of $V(G)$ such that



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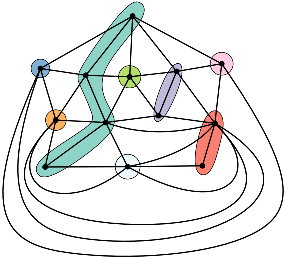
- Each part of P induces a *geodesic* (shortest path) in G; and



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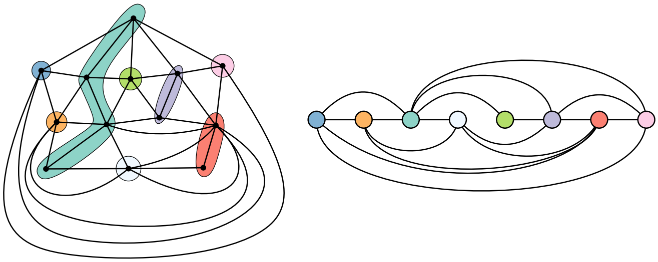
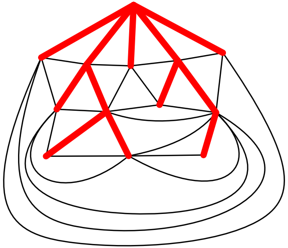
- Each part of P induces a *geodesic* (shortest path) in G; and
- The quotient graph $H := G/\mathcal{P}$ has treewidth at most 8



Layered H-Partitions

Theorem (Dujmović-Joret-Micek-M-Ueckerdt-Wood 2019):
For any planar triangulation G and any *breadth-first spanning-tree* T of G, there exists a partition $\mathcal{P}$ of $V(G)$ such that



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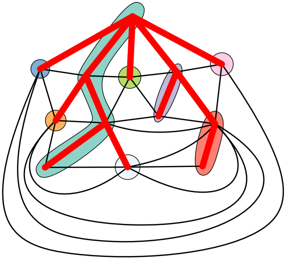
- Each part of P induces a *vertical path* in T



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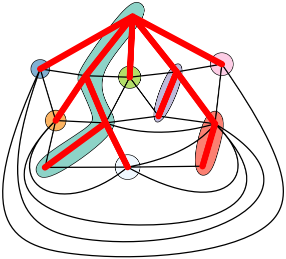
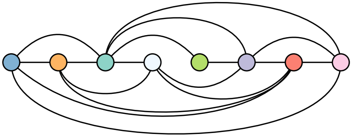
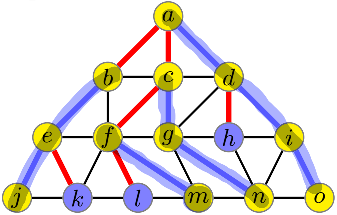
- The quotient graph $H := G/\mathcal{P}$ has treewidth at most 8

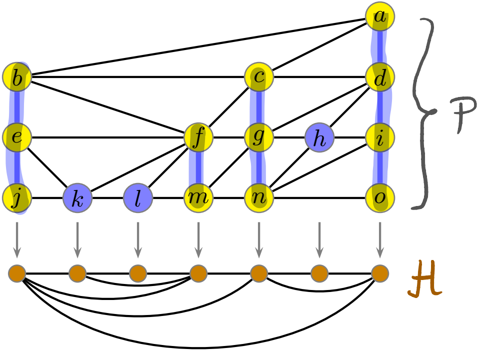




Equivalence :



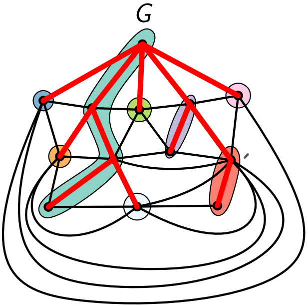
vertical paths

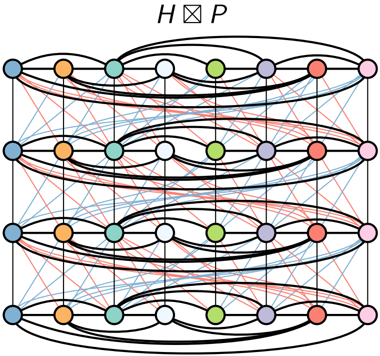


$G \subseteq H \boxtimes P$

Equivalence

Vertical H-partition theorem and product structure theorem are equivalent:





Equivalence

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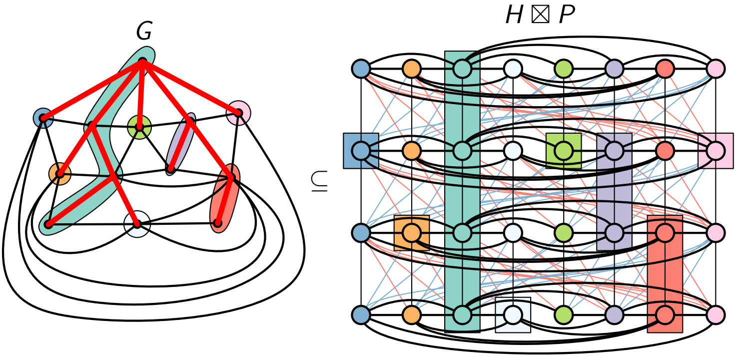
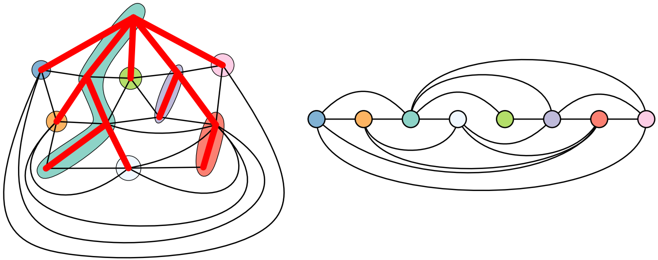


Layered H-Partitions

Theorem (Dujmović-Joret-Micek-M-Ueckerdt-Wood 2019):
For any planar triangulation G and any *breadth-first spanning-tree* T of G, there exists a partition $\mathcal{P}$ of $V(G)$ such that

- Each part of P induces a *vertical path* in T
- The quotient graph $H := G/\mathcal{P}$ has treewidth at most 8

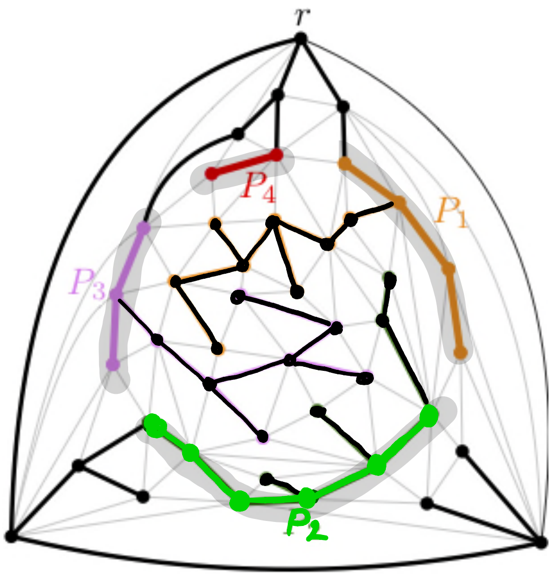


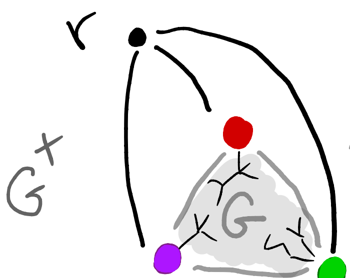
Proof: Partitioning Planar Graphs

Key lemma. Suppose

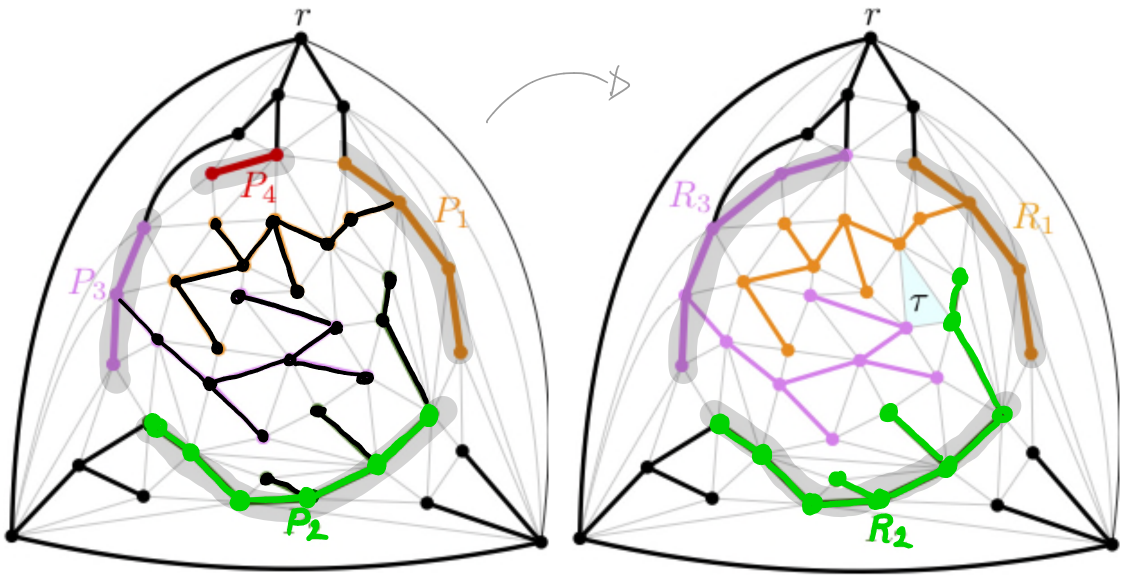
- G^+ plane triangulation
- T rooted spanning tree of G^+ with root on outer-face
- cycle C partitioned into vertical paths $P_1, \ldots, P_k$, with $k \leq 6$
- G near-triangulation consisting of C and everything inside.

Then G has a partition $\mathcal{P}$ into vertical paths where $P_1, \ldots, P_k \in \mathcal{P}$ s.t. $= G/\mathcal{P}$ has a tree-decomposition in which every bag has size at most 9 and some bag contains all vertices corresponding to $P_1, \ldots, P_k$.

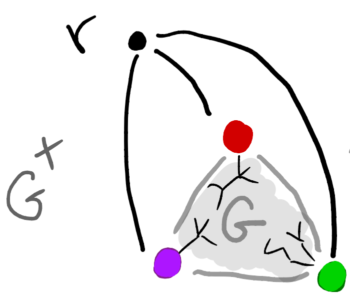




at the begining



Sperner lemma



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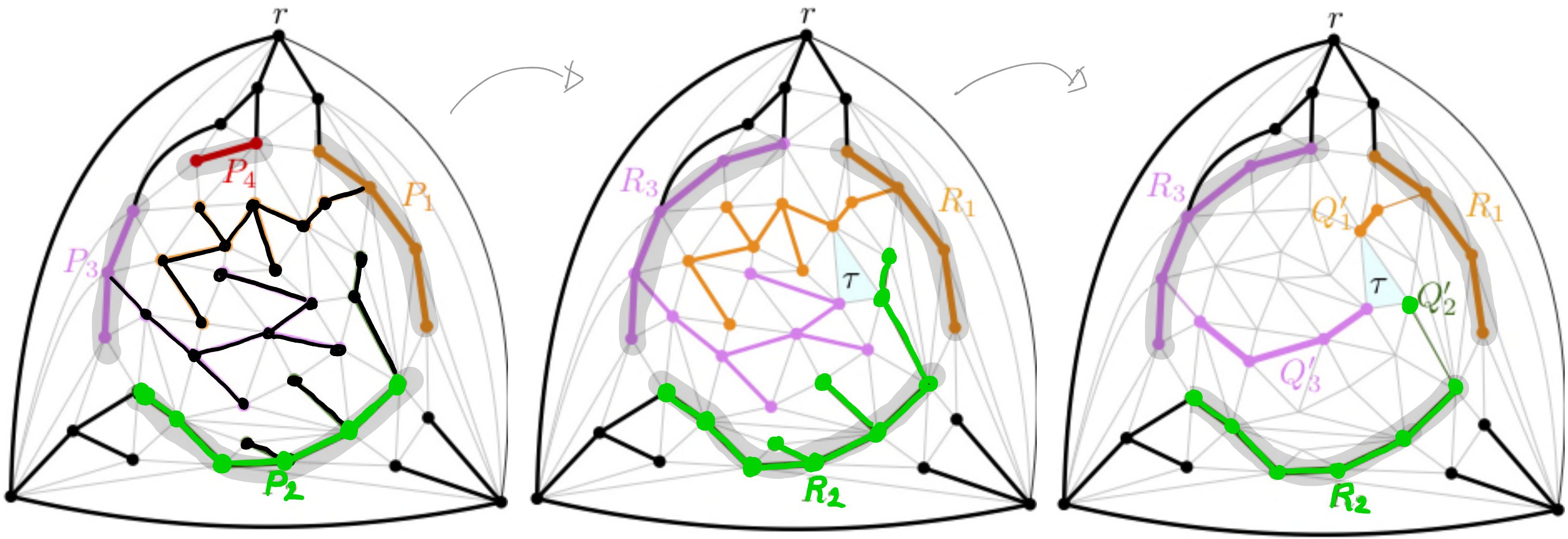

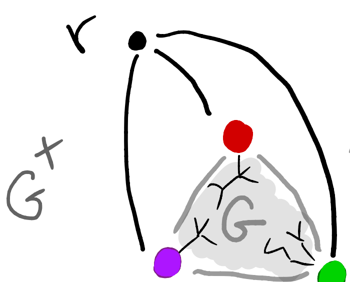

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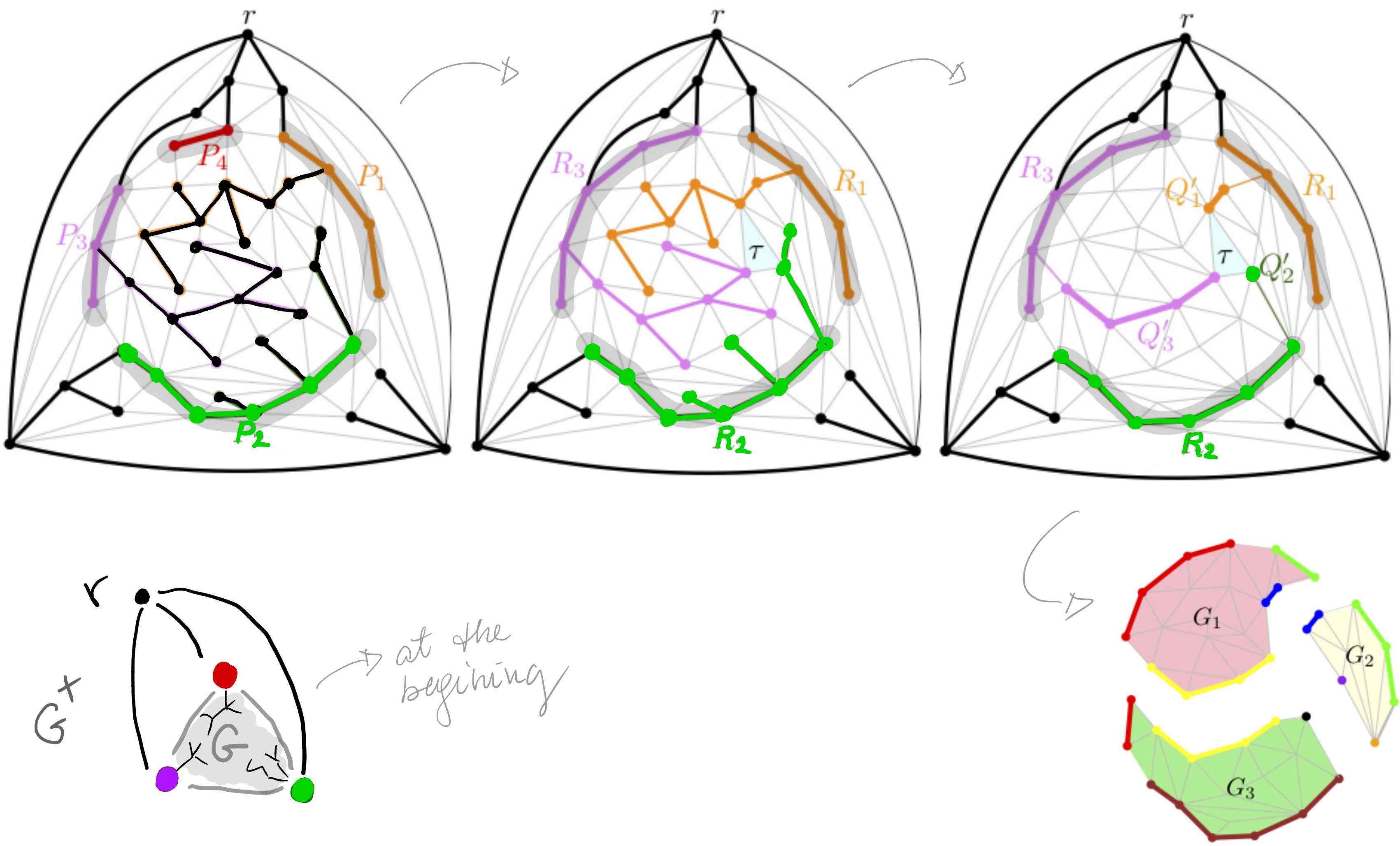
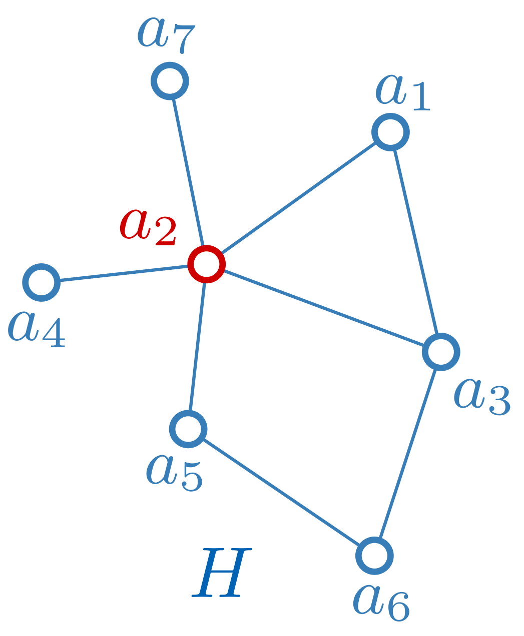

r
P4
P1
P3
P2
r
R3
R1
τ
R2
r
R3
Q'1
R1
τ
Q'2
Q'3
R2
r
G+
G
at the begining
G1
G2
G3

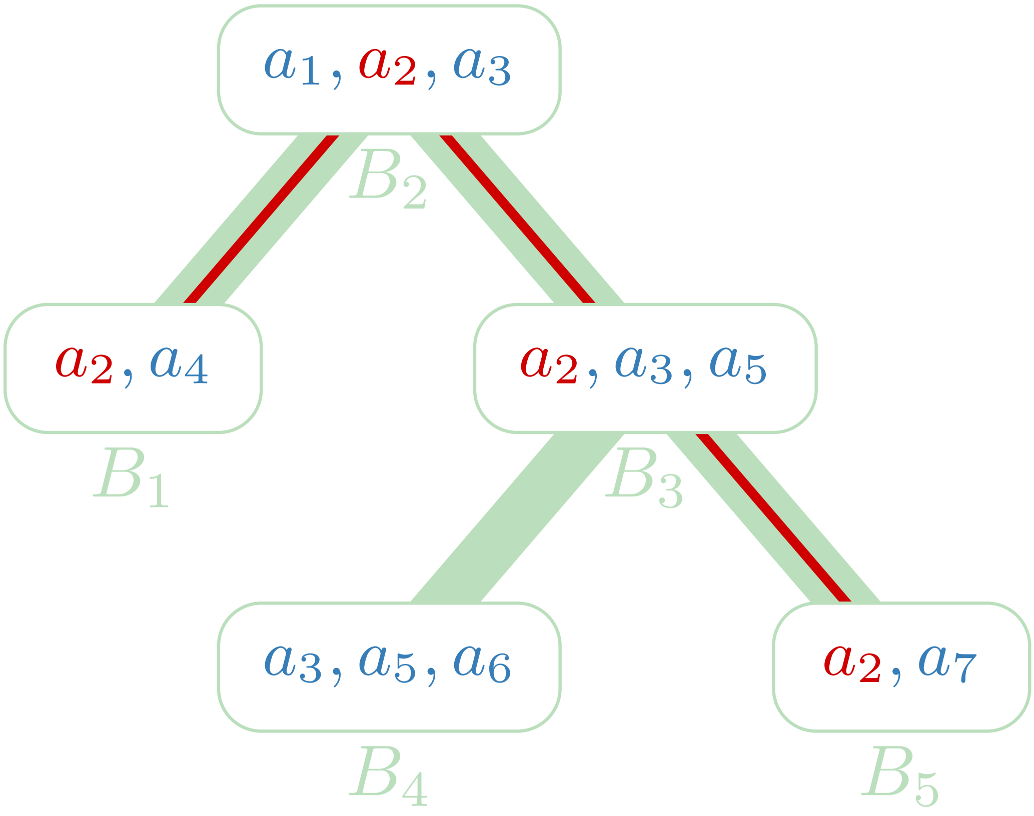
Tree Decompositions

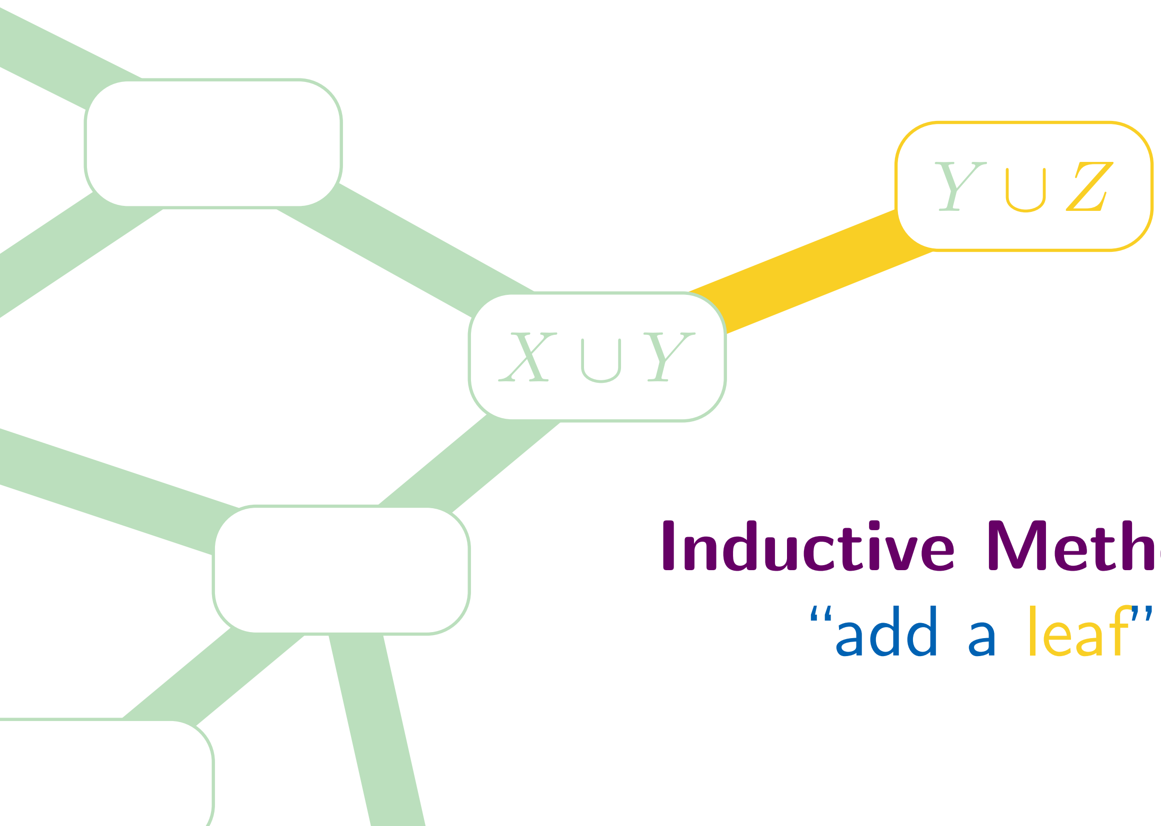
A tree decomposition of H are vertex sets (bags) $B_1, B_2, \ldots$ such that

- $B_1, B_2, \ldots$ are the vertices of a tree
- $v \in V(H) \Rightarrow \{B_i \mid v \in B_i\}$ subtree
- $uv \in E(H) \Rightarrow \exists i : u, v \in B_i$

The width is the maximum size of a bag -1.







Inductive Method 1
"add a leaf"

Inductive Method 2
"add a root"