Part 1: Product Structure Theory

$G \subseteq H \boxtimes P$

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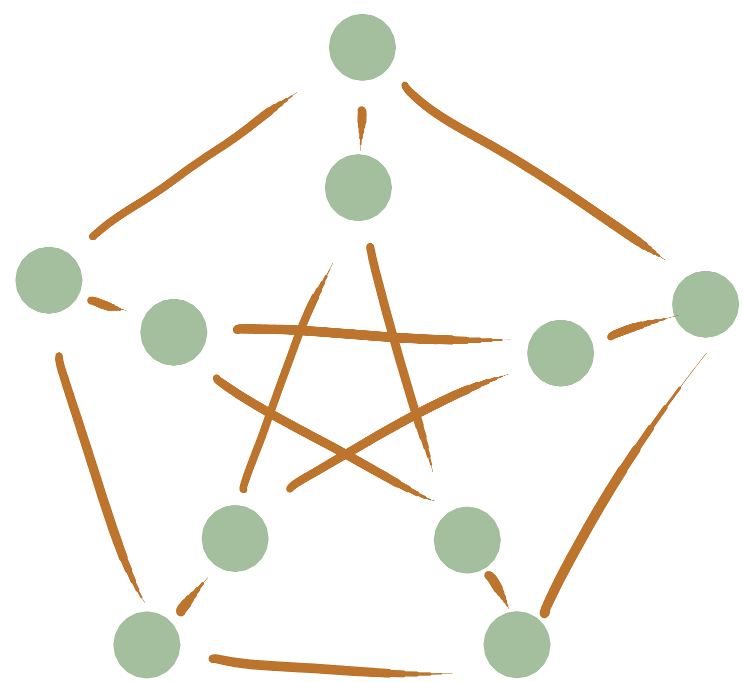
Problem 1

start via old slides

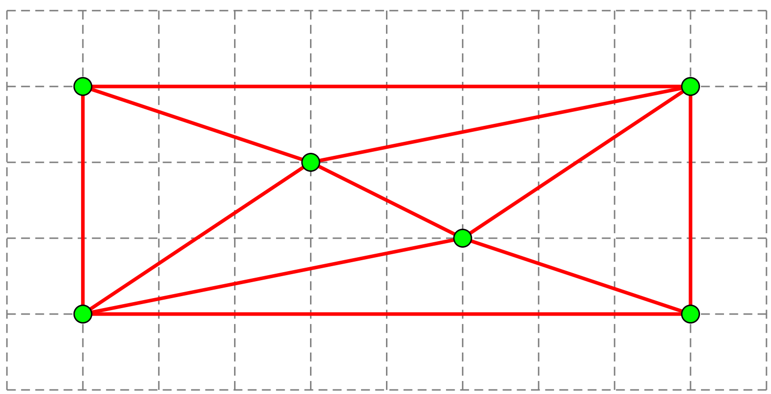
How to draw a graph (nicely)?

- edges are straight-line segments
- few edge crossings
- few bends per edge
- small area/volume
- symmetry
- resolution

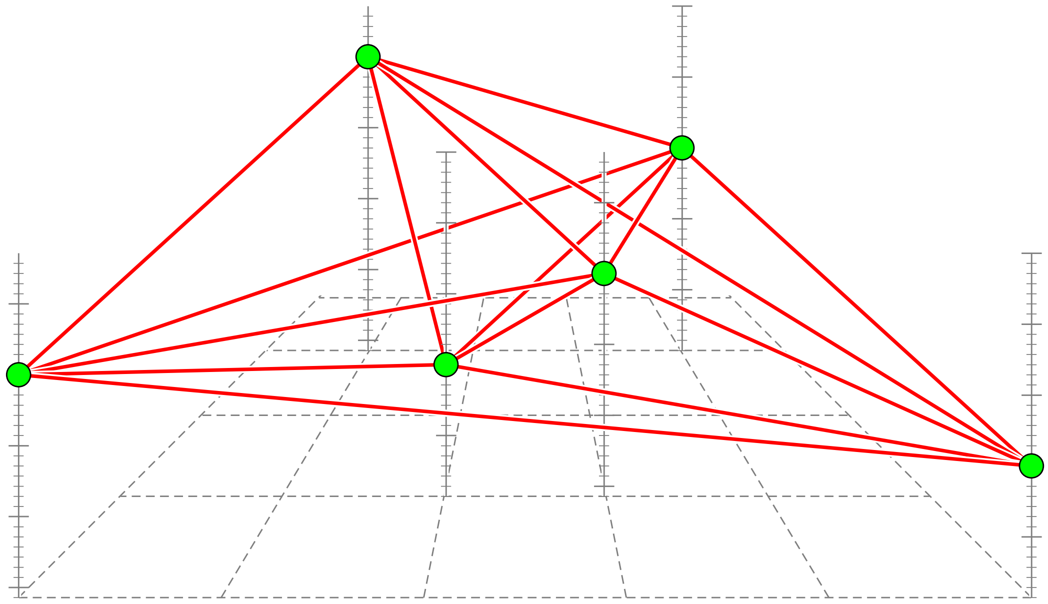


2-D and 3-D Straight-Line Grid Drawings

- vertices $\longrightarrow$ grid-points in $\mathbb{Z}^2$ ($\mathbb{Z}^3$)
- edges $\longrightarrow$ straight line segments
- no edge crossings



2-D



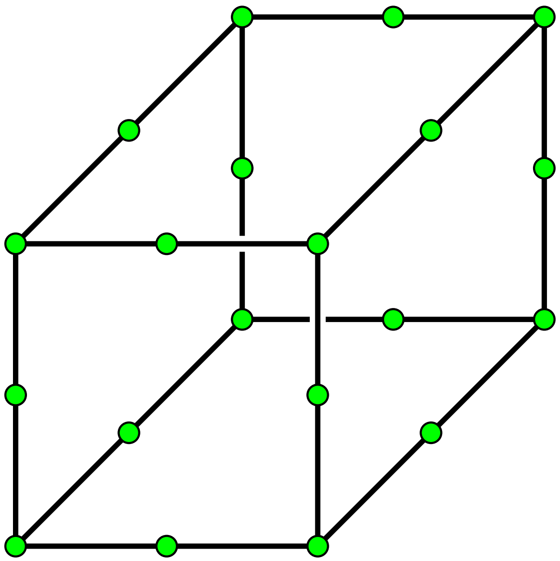
3-D

2-D and 3-D Straight-Line Grid Drawings

Main aesthetic criterion:

- 2-D: small area
- 3-D: small volume (of bounding box)

Measuring the volume of a box:



$\mathbf{3} \times \mathbf{3} \times \mathbf{3}$ box with volume $\mathbf{27}$

MOTIVATION: Planar graphs

Th: [de Fraysseix, Pach, Pollack '90, Schnyder '89]

PLANAR GRAPHS HAVE $\underbrace{\Theta(n) \times \Theta(n)}_{\Theta(n^2) \text{ volume}}$ 2D GRID DRAWINGS

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Q: [Felsner, Liotta, Wismath '01]

Can we do better in 3D?

$\hookrightarrow O(n)$

$\hookrightarrow o(n^2)$

FOLKLORE th: $\forall$ graph has a 3D grid drawing.

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Proof: Place vertex i at (i, i^2, i^3).

If edges $i_1 i_2$ and $i_3 i_4$ cross, then i_1, i_2, i_3, i_4 are coplanar

$$\det \begin{pmatrix} 1 & i_1 & i_1^2 & i_1^3 \\ 1 & i_2 & i_2^2 & i_2^3 \\ 1 & i_3 & i_3^2 & i_3^3 \\ 1 & i_4 & i_4^2 & i_4^3 \end{pmatrix} = \prod_{1 \leq \alpha < \beta \leq 4} (i_\alpha - i_\beta) = \emptyset$$

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Vandermonde matrix

$$\det \begin{pmatrix} 1 & i_1 & i_1^2 & i_1^3 \\ 1 & i_2 & i_2^2 & i_2^3 \\ 1 & i_3 & i_3^2 & i_3^3 \\ 1 & i_4 & i_4^2 & i_4^3 \end{pmatrix} = \prod_{1 \le \alpha < \beta \le 4} (i_\alpha - i_\beta) \neq 0$$

QED

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QED

- vertex i at $(i, i^2 \bmod p, i^3 \bmod p)$
  $\Rightarrow \Theta(n^3)$ volume Eades, Cohen, Lin, Ruskey '96
- Erdős discovered this trick in 2D in 1951

WHAT IS KNOWN?

GRAPH FAMILY	VOLUME	REFERENCE
K_n, arbitrary	$\Theta(n^3)$	Eades, Cohen, Lin, Ruskey '96
$O(1)$ colourable	$\Theta(n^2)$	Pach, Thiele, Toth '97
$O(1)$ max degree	$\Theta(n^{3/2})$	D. & Wood '04
$O(1)$ outerplanar series-parallel	$\Theta(n)$	– Felsner, Liotta, Wismath '01 – Di Giacomo, Liotta, Wismath '02
$O(1)$ treewidth	$\Theta(n)$	– D., Morin, Wood '05 – Wiechert '18

~~OPEN~~ PROBLEM

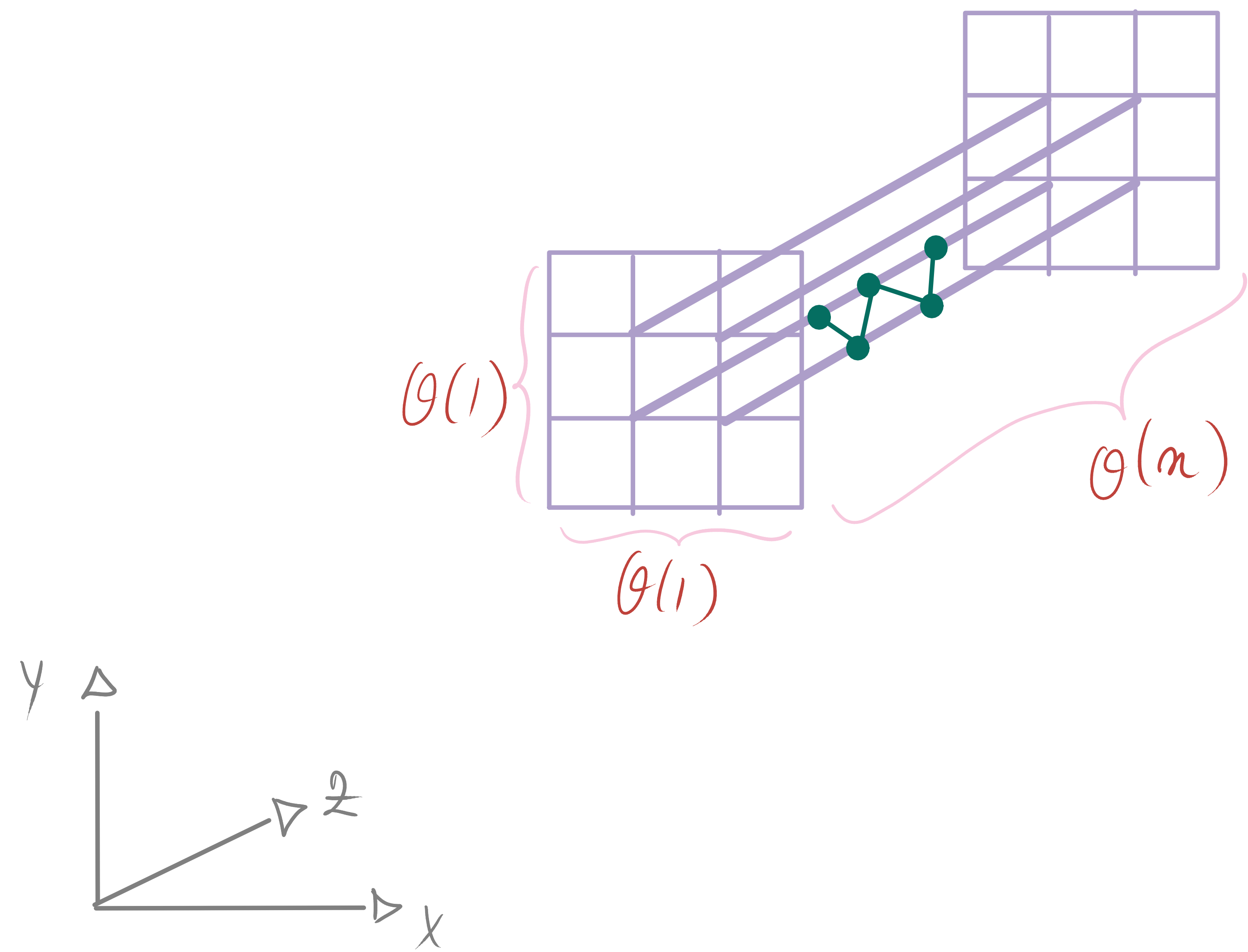
Q: [Felsner, Liotta, Wismath '01]

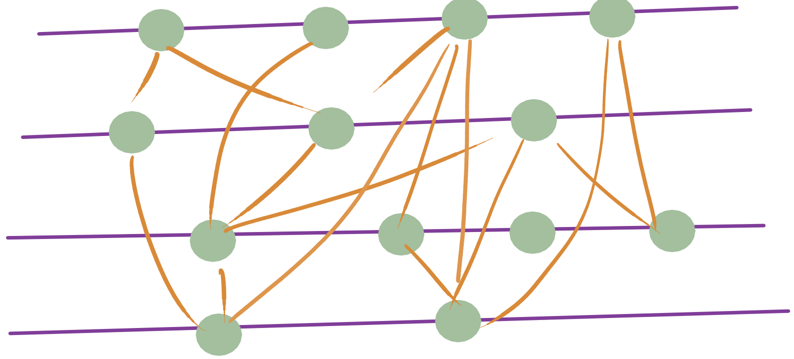
Do planar graphs have $\Theta(n)$ volume 3D grid drawings?

$o(n^2)$

planar: $\Theta(n^2) \to \Theta(n^{3/2}) \to \Theta(n \lg^c n) \to \Theta(n \lg n) \to \Theta(n)$

LINEAR VOLUME



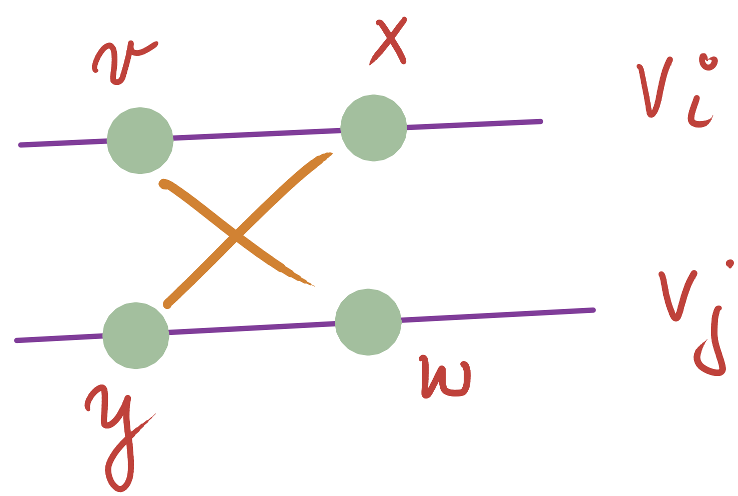


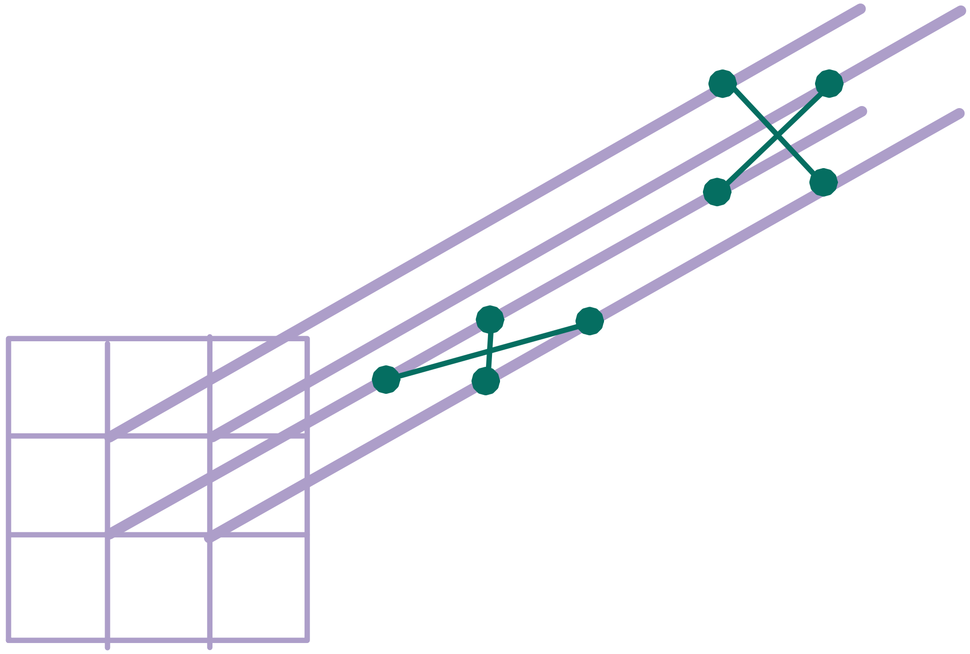
t-TRACK LAYOUT

- $\{V_1, V_2, \ldots, V_t\}$ – vertex colouring
- total order $<_i$ of each V_i (track)
- no X-crossing

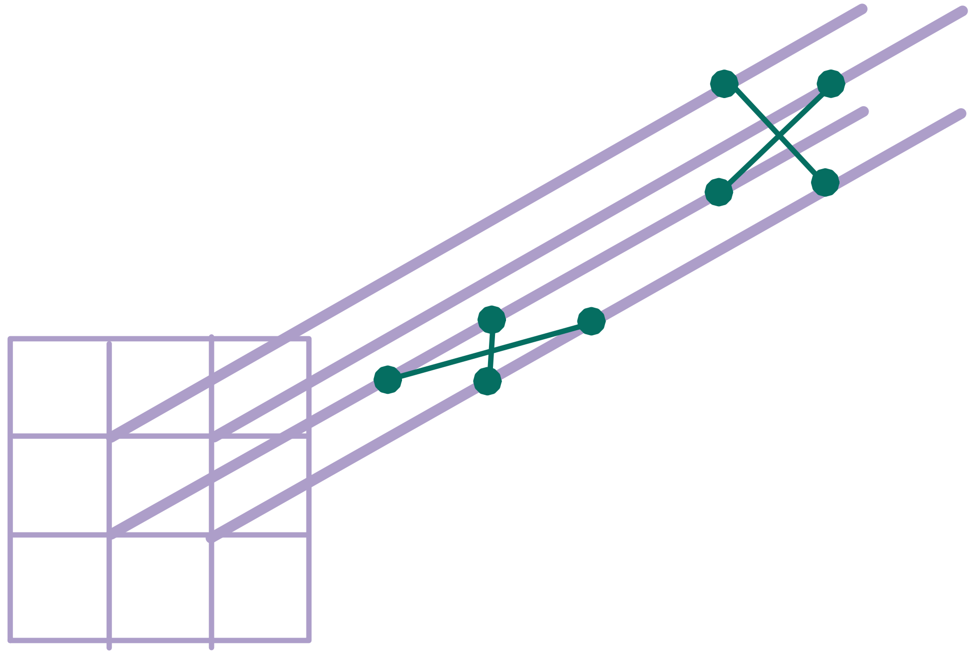
Def: X-crossing

edges vw, xy
s.t $v <_i x$ and $y <_j w$





why track layouts?

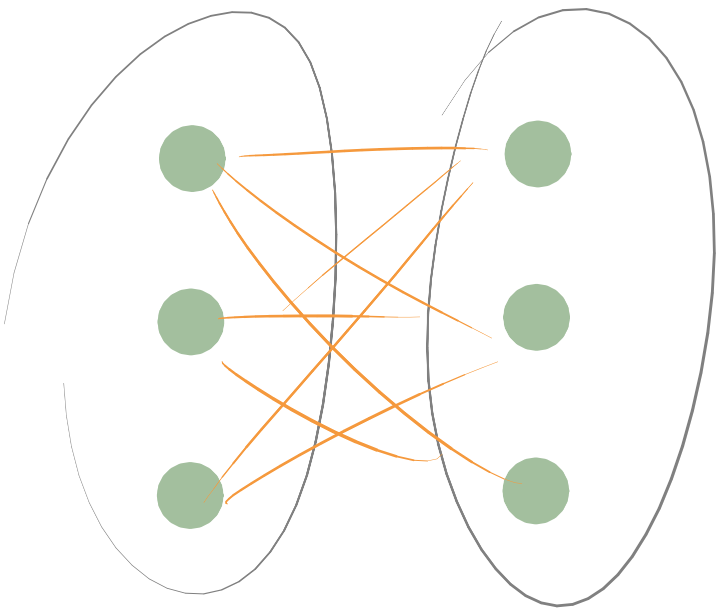


why track layouts?

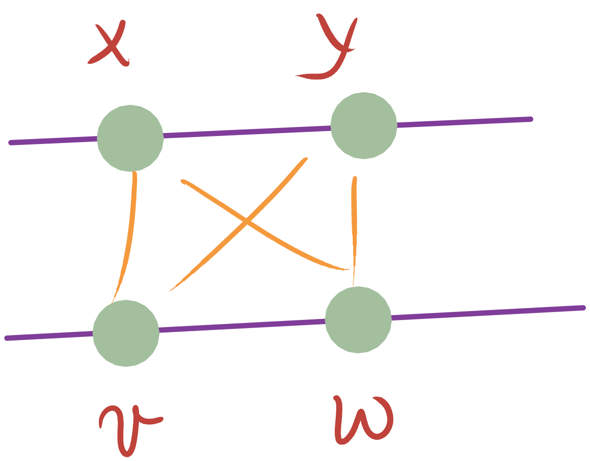
Thm: [D. & Wood '04 / D., Morin, Wood '05]
Every graph G with t-track layout has a 3D grid drawing in $O(t^2 \cdot n)$ volume.

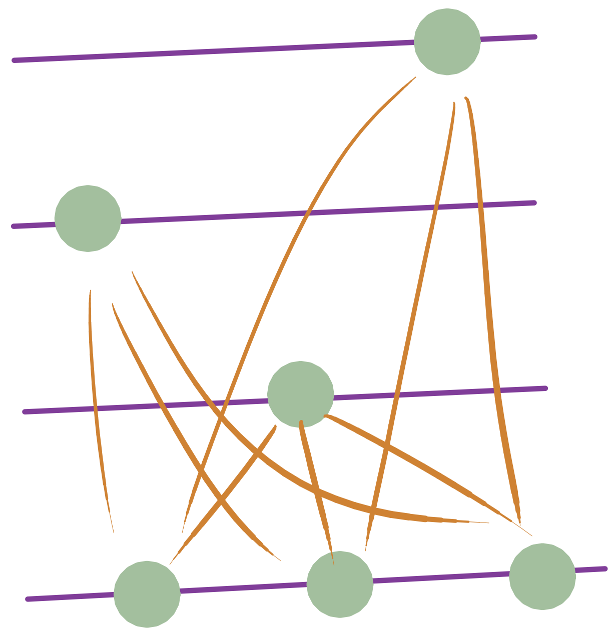
$O(x^7 \cdot t \cdot n)$

EXAMPLES: $K_{n,n}$

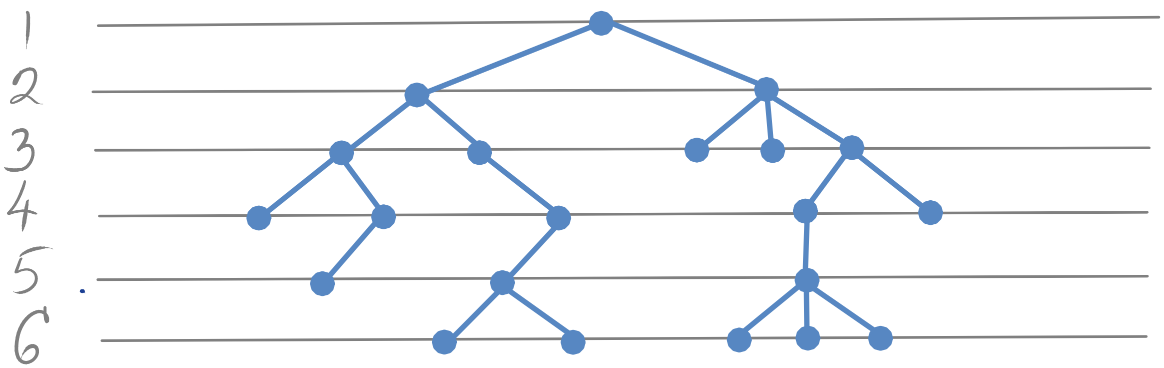


$$bn(K_{n,n}) = n + 1$$



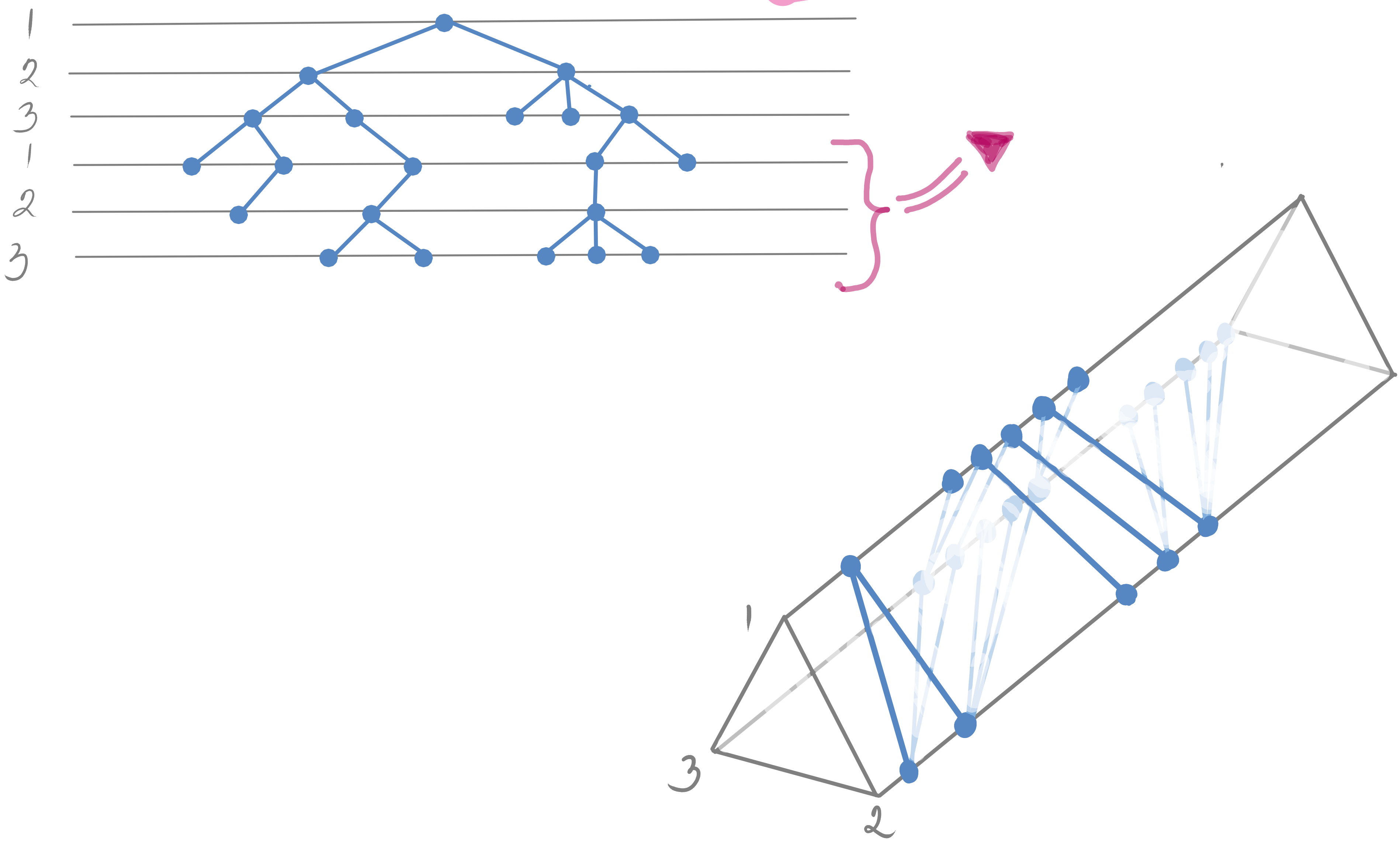


EXAMPLES: TREES



BFS

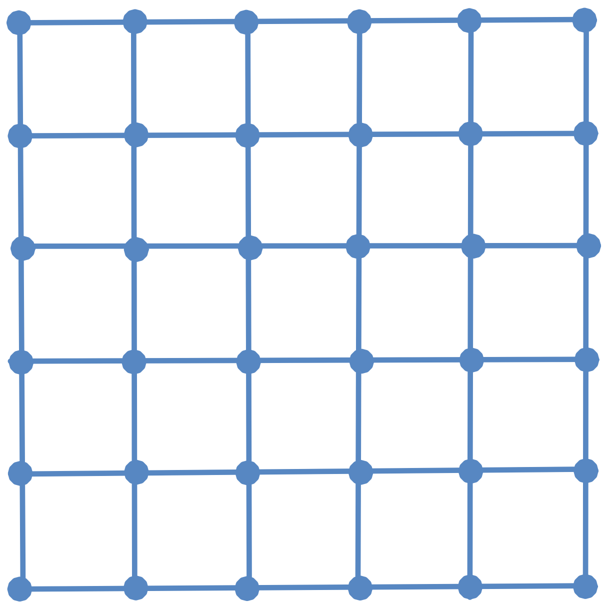
EXAMPLES: TREES



$tn(tree) \leq 3$

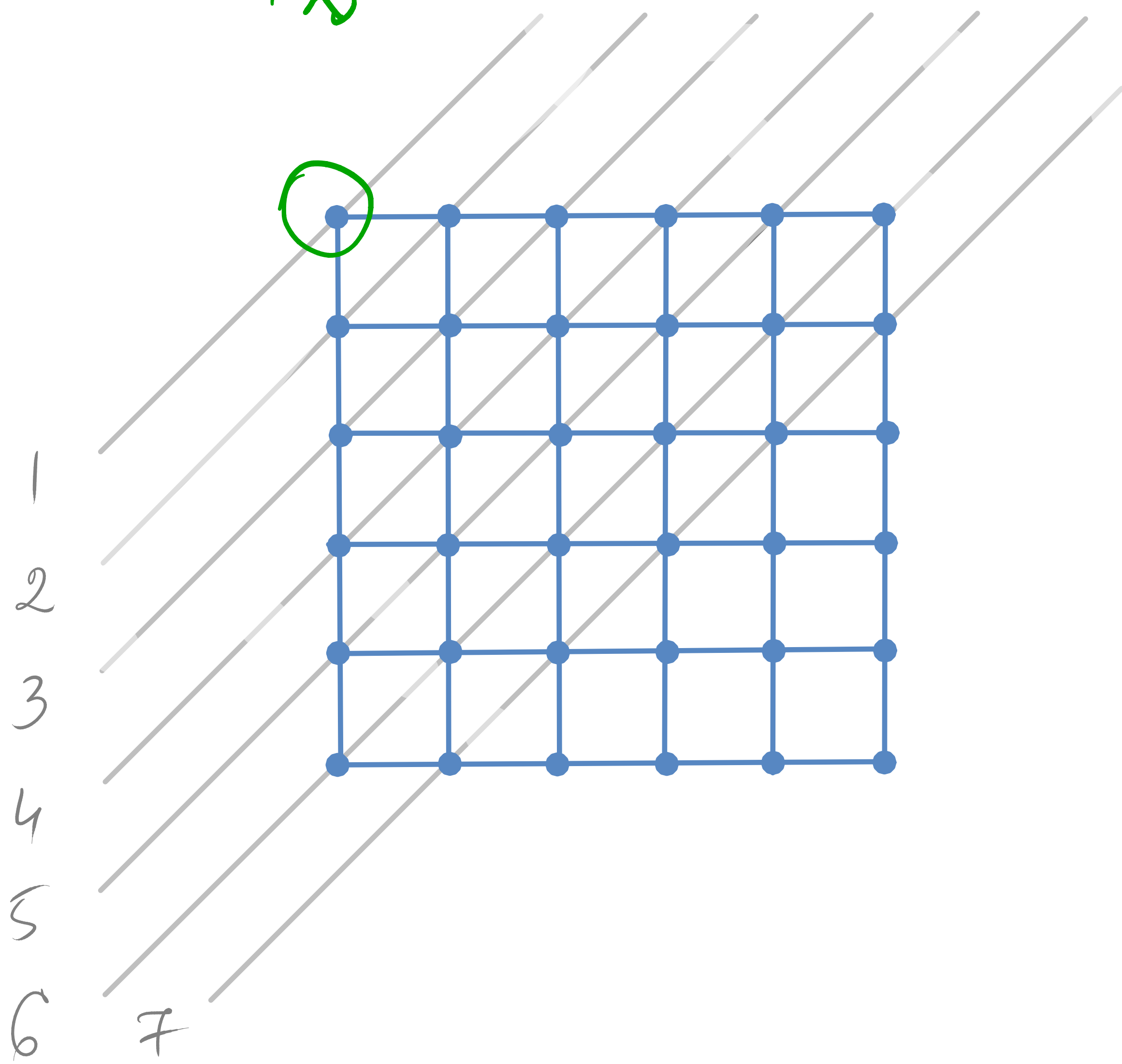
WRAPPING LEMMA

EXAMPLES: GRID



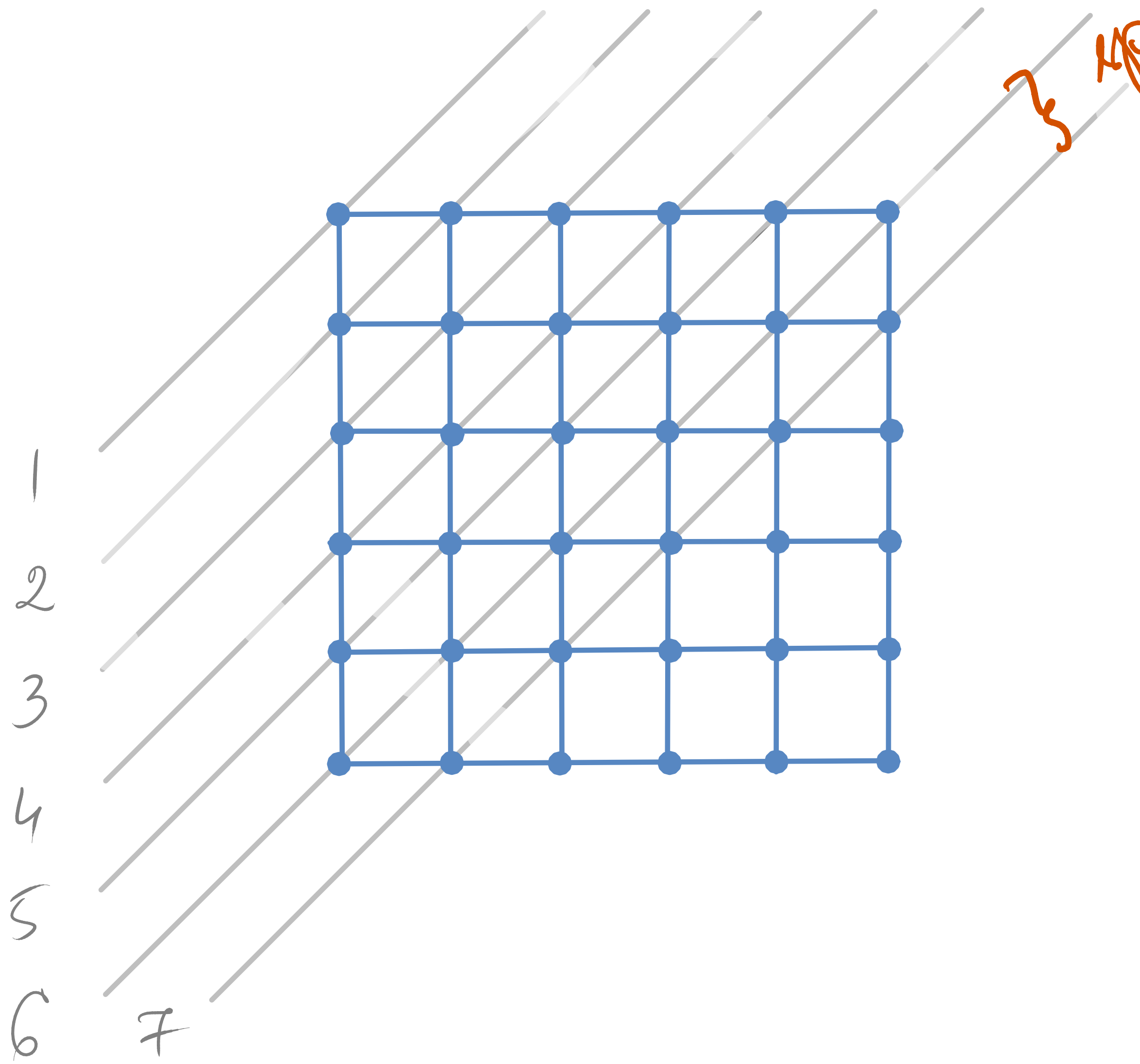
EXAMPLES: GRID

BFS again



EXAMPLES: GRID

7 краи =1

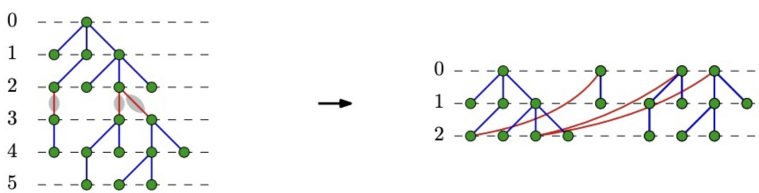


Span $\Rightarrow$ qn

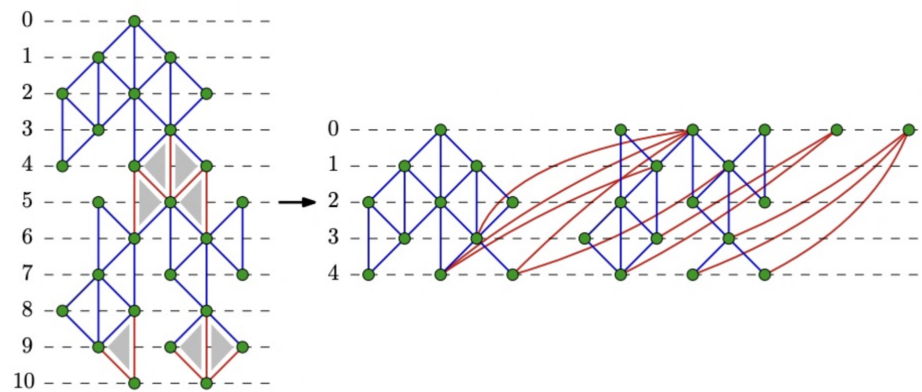
Fn track layout: $\{(V_i, <_i) : 1 \leq i \leq t\}$ with edge span $s \Rightarrow qn(G) \leq 2 \cdot s + 1$.

Span → tn

Thm track layout: $\{(V_i, <_i) : 1 \leq i \leq t\}$ with edge span s $\Rightarrow$ $tn(G) \leq 2 \cdot s + 1$.



Images by Sergey Pupyrev



HISTORY

Q1:

Do planar graphs have $O(1)$ track num?

$\to O(n)$ 3D grid drawing

HISTORY

Q_1:

Do planar graphs have $O(1)$ track num?

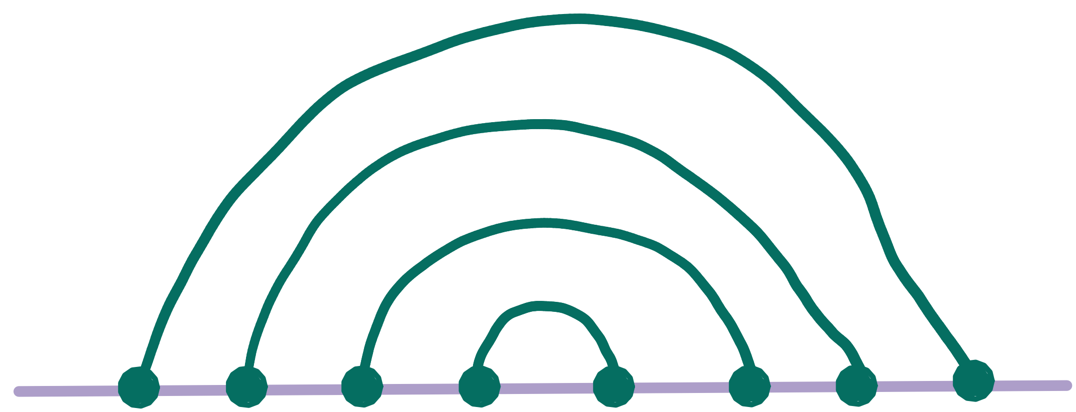
$\hookrightarrow O(n)$ 3D grid drawing

Q_2: [Heath, Leighton, Rosenberg: SICOMP, SIDMA '92]

Do planar graphs have $O(1)$ queue num?

$\hookrightarrow$ conjectured YES

$Q_1 = Q_2$



TRACK NUMBER

- $\Theta(\sqrt{n})$ was best known for 20+ years

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- [FOCS 2010, Di Battista, Frati, Pach]
  $O(\log^8 n)$
- [FOCS 2013, D., Morin, Wood] $O(\log n)$
- [2019 D., Joret, Micek, Morin, Ueckerdt, Wood] $O(1)$

TOOL : Product Structure

Ranking Graph Classes by Complexity

Simple

- paths (forests of paths)
- trees (forests)
- k-Trees (graphs of treewidth at most k)
- $\vdots$
- planar graphs
- $\vdots$
- proper-minor closed families
- $\vdots$
- bounded expansion
- $\vdots$
- all graphs

Complicated

Ranking Graph Classes by Complexity

Simple

- paths (forests of paths)
- trees (forests)
- k-Trees (graphs of treewidth at most k) "tree like"
- $\vdots$
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- $\vdots$
- proper-minor closed families
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Complicated

Ranking Graph Classes by Complexity

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- $\vdots$
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- $\vdots$
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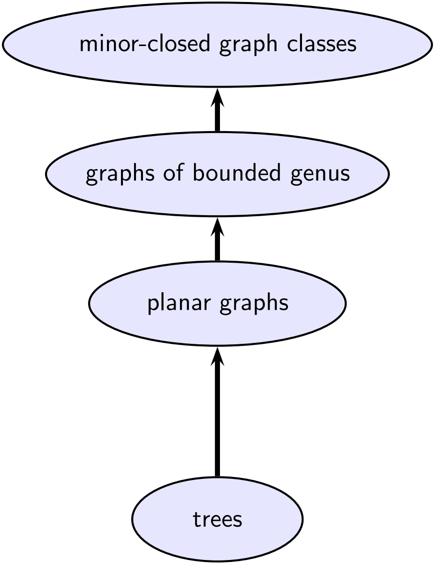
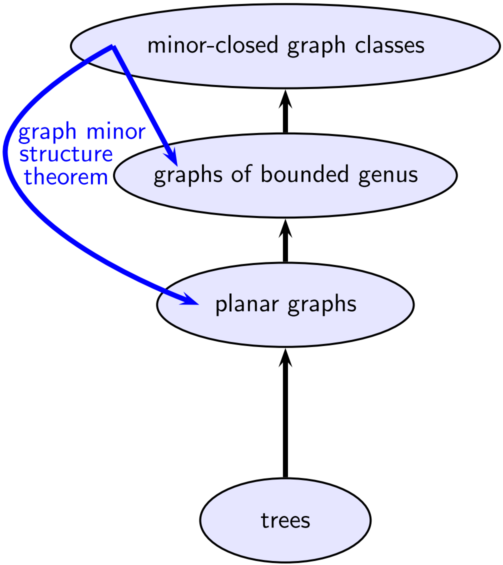
Complicated

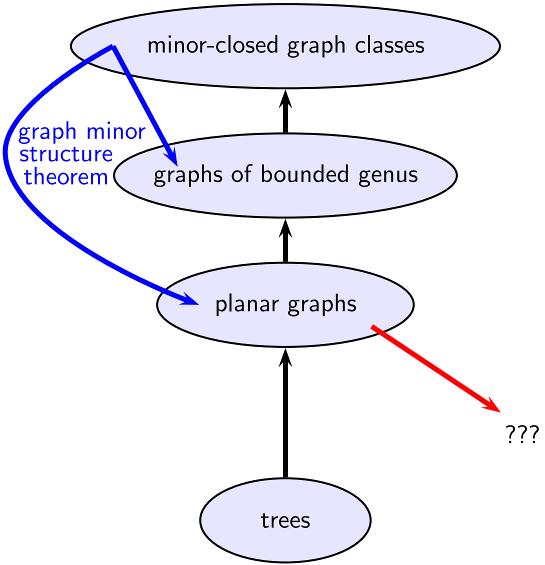
minor-closed classes



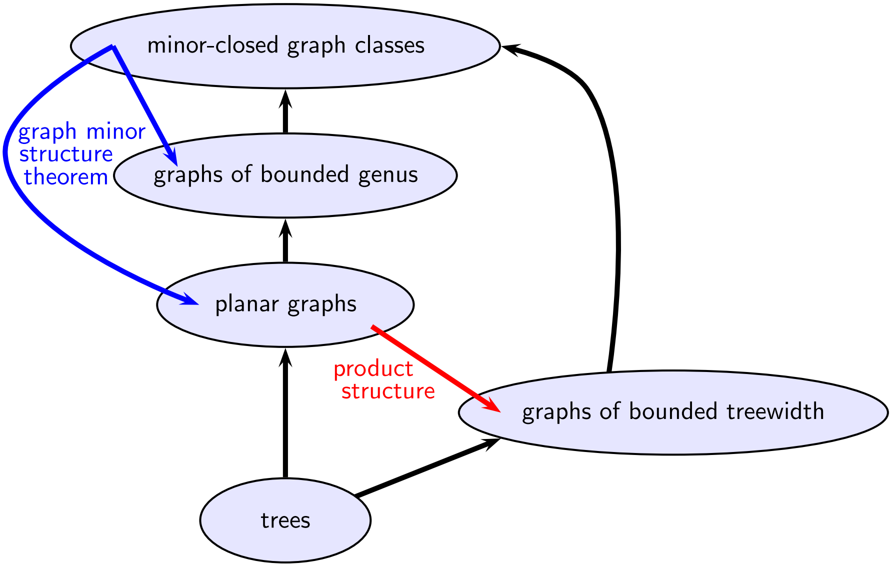
graph minor structure theorem



structure of planar graphs

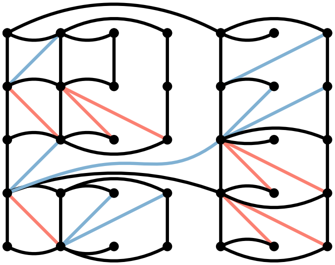


product structure



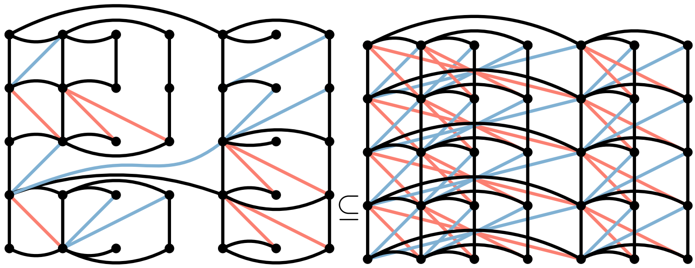
Informally

- Can we *factor* a planar graph into simpler graphs?



structure of planar graphs

theorem [Dujmović, Joret, Micek, Morin, Ueckerdt, Wood '19]
every planar graph G is a subgraph of $H \boxtimes P$
for some graph H with treewidth $\leqslant 8$ and some path P



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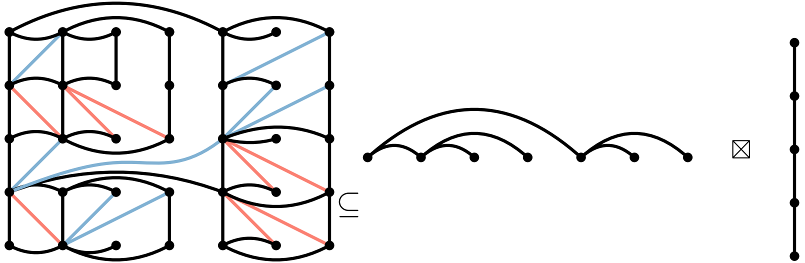

strong product

For two graphs A and B, the *strong product* $A \boxtimes B$ is a graph:

- $V(A \boxtimes B) := V(A) \times V(B)$
- (a_1, b_1) and (a_2, b_2) are adjacent if and only if:
  - $a_1 = a_2$ and $b_1 b_2 \in E(B)$;
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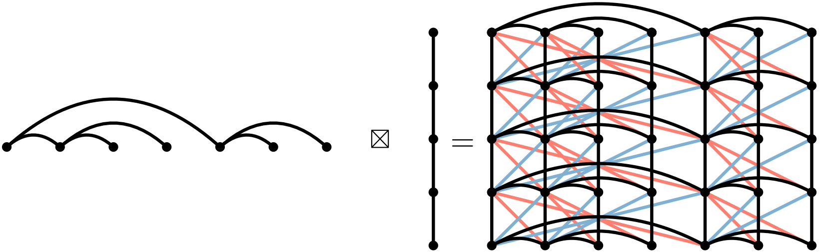

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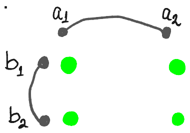

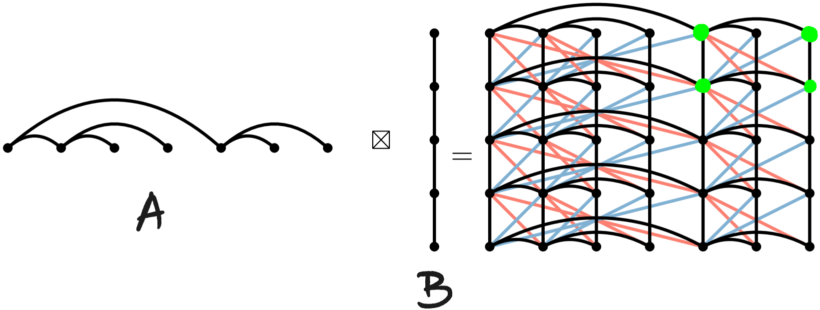

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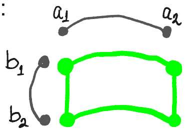
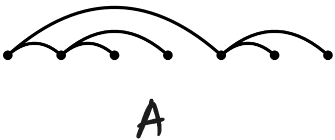
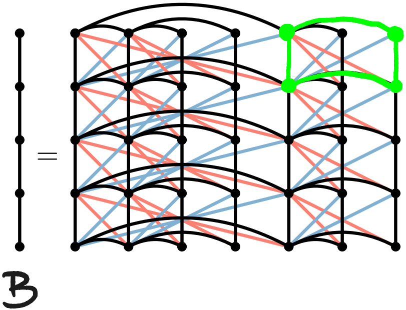




$\boxtimes$



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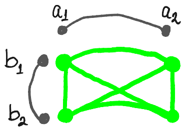
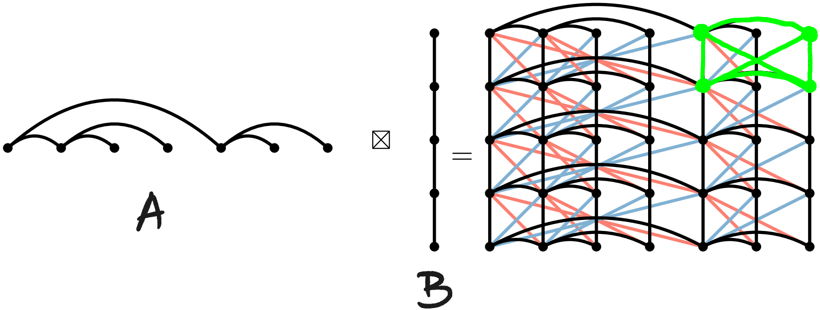
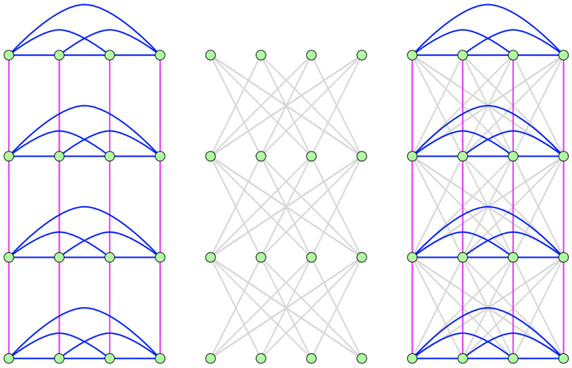




cartesian, direct, strong product



(a) $P_4 \square K_4$ (b) $P_4 \times K_4$ (c) $P_4 \boxtimes K_4$

Fig. 4: Examples of graph products: (a) cartesian, (b) direct, (c) strong.

structure of planar graphs

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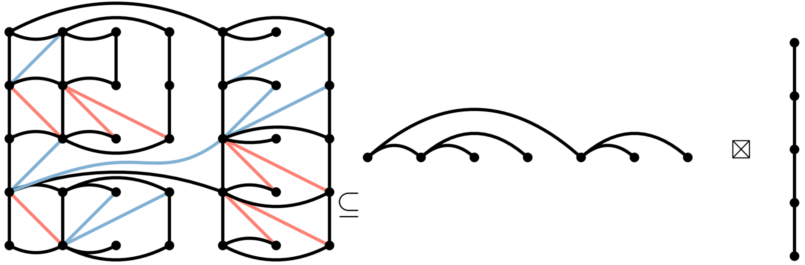
for some graph H with treewidth $\leqslant 8$ and some path P

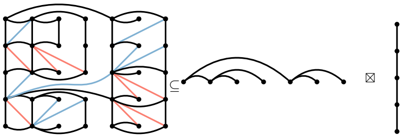


what is it good for?

Why?

$G \subseteq H \boxtimes P$

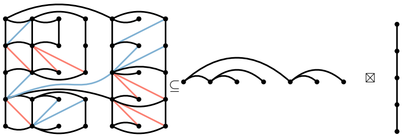
- H is a graph of treewidth at most 8



Why?

$G \subseteq H \boxtimes P$

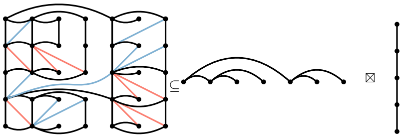
- H is a graph of treewidth at most 8
- Many problems are easy for H



Why?

$G \subseteq H \boxtimes P$

- H is a graph of treewidth at most 8
- Many problems are easy for H
- Extending a solution from H to $H \boxtimes P$ is sometimes easy



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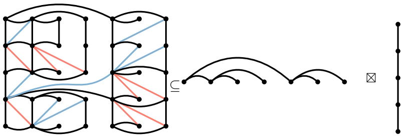
- Examples:



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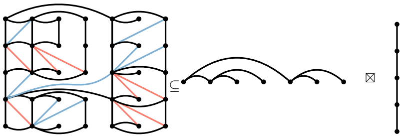
- Examples:
  - queue number 1992, 3D grid drawings → 2002, track number



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- H is a graph of treewidth at most 8
- Many problems are easy for H
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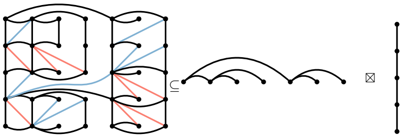
- Examples:
  - queue number
  - nonrepetitive colouring — 2002



Why?

$G \subseteq H \boxtimes P$

- H is a graph of treewidth at most 8
- Many problems are easy for H
- Extending a solution from H to $H \boxtimes P$ is sometimes easy
- Examples:
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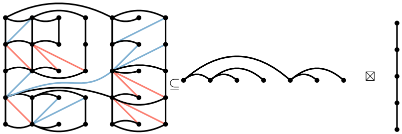
  - nonrepetitive colouring
  - p-centered colouring



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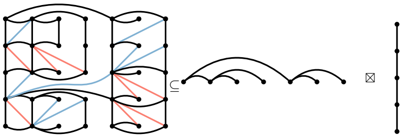

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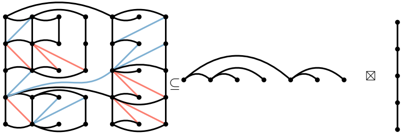

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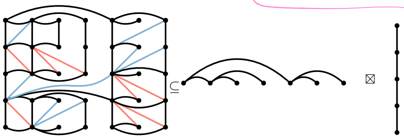
Clustered Hadwiger



Back to track number
of planar graphs